

# Energy Momentum Tensor and Operator Product Expansion in Local Causal Perturbation Theory

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ABSTRACT. We derive new examples for algebraic relations of interacting fields in local perturbative quantum field theory. The fundamental building blocks in this approach are time ordered products of free (composed) fields. We give explicit formulas for the construction of Poincaré covariant ones, which were already known to exist through cohomological arguments.

For a large class of theories the canonical energy momentum tensor is shown to be conserved. Classical theories without dimensionful couplings admit an improved tensor that is additionally traceless. On the example of  $\varphi^4$ -theory we discuss the improved tensor in the quantum theory. Its trace receives an anomalous contribution due to its conservation.

Moreover we define an interacting bilocal normal product for scalar theories. This leads to an operator product expansion of two time ordered fields.

ZUSAMMENFASSUNG. Im Rahmen der kausalen Störungstheorie leiten wir neue Beispiele für algebraische Relationen wechselwirkender Quantenfelder her. Wir geben explizite Formeln für die Konstruktion Poincaré kovarianter zeitgeordneter Produkte freier (zusammengesetzter) Felder an, welche in diesem Zugang die Grundbausteine bilden. Bisher war nur deren Existenz aufgrund kohomologischer Argumente bekannt.

Für eine große Klasse von Theorien zeigen wir die Erhaltung des kanonischen Energie-Impuls-Tensors. Für klassische Theorien, die keine dimensionsbehafteten Kopplungen enthalten, existiert ein verbesserter Tensor, der zusätzlich spurfrei ist. Am Beispiel der  $\varphi^4$ -Theorie untersuchen wir diesen Tensor in der Quantentheorie. Als Folge der Erhaltung bekommt die Spur anomale Beiträge.

Darüberhinaus geben wir die Definition eines bilokalen wechselwirkenden Normalproduktes für skalare Theorien. Mithilfe des Normalproduktes finden wir die Operator-Produkt-Entwicklung für das zeitgeordnete Produkt zweier Felder.



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## CHAPTER 1

### Introduction

*“There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable ... ”*<sup>1</sup>

Today quantum field theory (QFT) provides the best unification of classical relativistic field theory with the axioms of quantum mechanics. During the last decades a high agreement between theoretical predictions and observed data was achieved. The theoretical results referring to experimental data from scattering processes are usually derived via perturbation theory around the free quantum field or around the classical field theory. In these regimes either the coupling or Planck’s constant can be regarded as a small parameter and therefore perturbation theory seems to apply as a suitable tool.

On the other hand there is no hope to derive any realistic statement about the strong coupling regime of elementary particle physics from the perturbative point of view. This subject is addressed in the formulation of QFT on the lattice where spacetime becomes discrete. In the last years especially lattice gauge theory benefits from the increasing computing power that is available to describe the structure of hadrons.

A quite severe drawback in the formulation of QFT is the fact that up to now no realistic model of an interacting non perturbative and continuous theory exists in four dimensions. Therefore the perturbative approach has become the most popular one, (see e.g. [IZ85]). The quantization of free fields as an operator relation following the principles of quantum mechanics and special relativity constitutes the starting point. The interaction is introduced as a perturbation and the coupling is treated as an expansion parameter for the interacting Green’s functions. These functions show two different divergencies, ultraviolet and infrared ones. The former ones reflect the distributional character of the free field operators whereas the latter ones originate from the long range forces carried by massless particles. These divergencies are removed in the various processes of renormalization, of which we mention only the most general one given by BPHZ(L). Unfortunately, all investigations concerning the convergence of the perturbation series yield a negative result.

In contrast to the perturbative formulation of quantum field theory which relies on the existence of some specific models, an axiomatic approach to quantum field theory (also called algebraic QFT) was given by Haag and Kastler

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<sup>1</sup>to be continued on page 85.

[Haa92][and references there]. In their approach the starting point is the existence of a net of *local* algebras of observables. The principle of locality is taken into account by the requirement that space like separated observables commute reflecting the fact that measurements on space like separated experiments can not influence each other. Under the requirement that the Poincaré group is unitarily implemented and isotony holds the net is completely fixed. A representation of this abstractly defined algebra by operators on a Hilbert space is generated through the GNS-construction provided one has defined (physically relevant) states, namely positive normed linear functionals.

Neither the existence of quantum fields nor of particles serves as an input and their correlation still is a subject of research [BH00]. Especially the treatment of non perturbative and model independent effects, like the relation between spin & statistics turn out to be customary applications of algebraic QFT.

An adapted version of perturbative quantum field theory that fits nicely into the algebraic framework was derived by Bogoliubov, Shirkov [BS76] and Epstein-Glaser [EG73]. Their approach focuses on the construction of a local  $S$ -matrix as a formal power series of a compactly supported coupling by the axioms of causality and Poincaré covariance mainly. This  $S$ -matrix gives rise to a local observable in the algebraic sense [BF96], namely the relative  $S$ -matrix. From this functional interacting (composed) fields are derived which serve as a basis for local observables constituting the local algebras. By the principles of locality, the  $S$ -matrix only has to be known in the region in which the interacting algebras live and on which the coupling is kept fixed. No IR-divergencies related to an infinite range of interaction appear. Moreover the inductive construction takes care of the UV-renormalization in a very elegant way. The limit in which the coupling becomes a constant over all spacetime like in the usual approach is referred to as the adiabatic limit. Since all objects are defined by formal power series the question of convergence is not addressed in that framework.

Local perturbation theory was applied successfully to QED [Sch95][and references herein]. Blanchard and Seneor [BS75] have shown that the adiabatic limit for Green's functions and for Wightman functions exists for QED and also for massless  $\varphi^4$ -theory. Epstein and Glaser already proved the existence of the adiabatic limit in massive theories [EG73].

It was shown by Dütsch [Düt97] that the Epstein-Glaser definition of Green's functions yields the Gell-Man-Low formula in that limit. But for non Abelian gauge theories the situation is worse. Therefore one tries to avoid the adiabatic limit in general. In [BF00] Brunetti and Fredenhagen have shown that a variation of the interaction outside the localization region only acts as a unitary transformation on the interacting fields. Moreover they have given a purely algebraic construction of the adiabatic limit.

The elementary building blocks for the  $S$ -matrices are the time ordered products of (composed) free fields, i.e. Wick monomials. In their inductive construction ambiguities generically appear through the process of extending distributions. Restrictions on these extensions are called *normalization conditions*. The requirement of Poincaré covariance forms the most important one. Stora and Popineau [SP82] and Dütsch et al. [DHKS94], [Sch95][chapter 4.5] have given a cohomological existence proof of such an extension. In [BPP99]



we presented an explicit extension in lowest order and gave a general result in [Pra99b] in form of an inductive construction.

Further normalization conditions have been given in [DF99]. They imply the interacting field equations. In [Boa99] Boas has given a generalization of these conditions such that they can be applied to derivative couplings, too.

In the perturbative construction of QFT's symmetries play a major role. In classical field theory any symmetry of the Lagrangian gives rise to a conserved current via the Noether procedure. In the corresponding QFT one tries to preserve as many of these symmetries as possible. The breaking of a classical symmetry in the quantum theory is called an anomaly. In [DF99] BRST invariance was shown to hold for QED and in [Boa99] the result was extended to non Abelian theories. The corresponding current conservation follows from a Ward identity for time ordered products involving one (free) current. These identities can be regarded as further normalization conditions.

This thesis focuses on two subjects. The first deals with translation invariance. In classical field theory the energy momentum tensor (EMT) is derived as the Noether current subject to a translation of the fields. In our situation translation invariance is broken through the coupling term and translation invariance only holds where the coupling is constant. We show that the same conservation equation can be maintained in local perturbative QFT for a quite general theory without derivative couplings. This equation is a consequence of a Ward identity which we prove with the methods developed in [DF99, Boa99]. Our result coincides with a similar investigation by Lowenstein [Low71] who also has shown the conservation of the canonical EMT for  $\varphi^4$ -theory in the framework of Zimmermann's normal product quantization [Zim71, Zim73a].

For classical theories which possess no dimensionful parameters an improved EMT exists which is also traceless [CCJ70]. The improved tensor is derived by addition of a conserved tensor, called improvement tensor. On the example of massless  $\varphi^4$ -theory we show that in the perturbative local formulation there still exists a conserved improvement tensor by a corresponding Ward identity. But the improved EMT inevitably produces the well known trace anomaly [CJ71] by virtue of the compatibility of both Ward identities. The anomaly is only defined up to a real parameter related to some normalization choice. Our result is in accordance with Zimmermann's analysis [Zim84] who also has given a derivation of the anomaly in terms of normal products.

The second focus is on the derivation of an operator product expansion in local perturbation theory. In [Zim71, Zim73b] Zimmermann has generalized his concept of local normal products to the case of bilocal normal products which allow for restricting the coordinates to the same value (the same way which is allowed in Wick or normal ordering of free fields). With the help of these objects he found an operator product expansion of the time ordered product of two scalar fields as suggested by Wilson [Wil69]. We give a similar approach here. The definition of a time ordered product with a bilocal Wick product in one entry allows for an operator product expansion of the time ordered product of the corresponding interacting fields. We perform the construction for  $\varphi^4$ -theory. With these interacting normal products we suggest a definition of a ground state in analogy to the free field case. We show that the corresponding

two point function is positive in the sense of formal power series as defined in [DF99].

The thesis is organized as follows. In chapter 2 we review the process of quantization of free scalar fields which are the fields that our thesis mainly deals with.

Chapter 3 introduces the notion of time ordered products. We use the formulation of Boas [Boa99] where auxiliary variables are used instead of Wick products. The inductive construction in the form [BF96, BF00] is explained. Then the normalization conditions [DF99, Boa99] follow. We cite the solution of the Poincaré covariant extension from [BPP99, Pra99b].

Interacting fields as formal power series according to Bogoliubov and Epstein-Glaser [BS76, EG73] are introduced in chapter 4.

The EMT is discussed in chapter 5. The canonical tensor is shown to be conserved locally for a quite general theory without derivative couplings. Its charge is shown to define the interacting momentum operator. The improved tensor is studied on the example of massless  $\varphi^4$ -theory. The trace anomaly is derived.

In the last chapter 6 we study the OPE for the time ordered product of two interacting scalar fields. The expansion follows from a suitable definition of a bilocal time ordered product.

At the end we give a conclusion and some outlook.

## CHAPTER 2

### Canonical quantization of free scalar fields

We review the process of quantization for the (bosonic) scalar free field. Our presentation of the subject follows the lecture notes [Fre99]. It emphasizes the possibility to define the algebra of the quantum field first in a purely algebraic manner. A representation of this algebra by (unbounded) operators on a Hilbert space is derived via the GNS-reconstruction once a state on the algebra is given. With the usual two point function defining a quasi free state, the Hilbert space becomes the usual symmetric Fock space.

Minkowski space is denoted by  $\mathbb{M}$  and the scalar product is  $x \cdot y = xy = \eta_{\mu\nu}x^\mu y^\nu$ .

#### 1. Free quantum field algebra

We consider the free scalar field  $\varphi$  of mass  $m$  which solves the Klein-Gordon equation:

$$D\varphi \doteq (\square + m^2)\varphi = 0. \quad (2.1)$$

As a hyperbolic differential equation the Klein-Gordon equation possesses a fundamental solution  $\Delta \in \mathcal{D}'(\mathbb{M})$  with causal support,  $\text{supp } \Delta \in \overline{V}_+ \cup \overline{V}_-$ , which solves the equation (in the weak sense):

$$\Delta(Df) = 0, \quad \forall f \in \mathcal{D}(\mathbb{M}) \quad (2.2)$$

and has the following initial values:

$$\Delta(0, \mathbf{x}) = 0, \quad \partial_0 \Delta(0, \mathbf{x}) = -\delta(\mathbf{x}). \quad (2.3)$$

Then any solution of the differential equation is completely determined by the initial values at time  $t$ :

$$f(t, \mathbf{x}) = \phi_0(\mathbf{x}) \quad \partial_0 f(t, \mathbf{x}) = \phi_1(\mathbf{x}), \quad (2.4)$$

according to

$$f(x^0, \cdot) = \Delta(x^0 - t) * \phi_1 + (\partial_0 \Delta)(x^0 - t) * \phi_0. \quad (2.5)$$

The distribution  $\Delta$  has a unique decomposition into advanced and retarded Green's functions:

$$\Delta = \Delta_{\text{ret}} - \Delta_{\text{av}}, \quad (2.6)$$

$$D\Delta_{\text{ret,av}} = \delta, \quad \text{supp } \Delta_{\text{ret}} \in \overline{V}_\pm, \quad (2.7)$$

The *algebra of observables*  $\mathfrak{A}$  is defined as an abstract algebra generated by  $\varphi(f), f \in \mathcal{D}(\mathbb{M})$  and the following conditions:

- I.  $f \mapsto \varphi(f)$  is linear,

- II.  $\varphi(Df) = 0$ ,
- III.  $\varphi(f)^* = \varphi(\overline{f})$ ,
- IV.  $[\varphi(f), \varphi(g)] = i \langle \Delta * g, f \rangle$ .

The  $*$ -operation is an algebra involution. The brackets  $\langle t, f \rangle = t(f)$ ,  $f \in \mathcal{D}(\mathbb{M})$  denote the evaluation of the functional  $t \in \mathcal{D}'(\mathbb{M})$  and the  $*$  means convolution. The algebra  $\mathfrak{A}$  is uniquely determined as a  $*$ -algebra with unit. If the supports of the test functions are contained in a bounded (usually causally complete) space time region  $\mathcal{O}$  one talks about the *local* algebra of observables  $\mathfrak{A}(\mathcal{O})$ .

The commutation relation is frequently written as

$$[\varphi(x), \varphi(y)] = i\Delta(x - y). \quad (2.8)$$

## 2. Representations of the observable algebras

A *state*  $\omega$  on the observable algebra  $\mathfrak{A}$  is a linear functional  $\omega : \mathfrak{A} \mapsto \mathbb{C}$  with the properties:

$$\omega(\mathbb{I}) = 1, \quad (2.9)$$

$$\omega(A^* A) \geq 0. \quad (2.10)$$

If a state on  $\mathfrak{A}$  is given one gets a representation of  $\mathfrak{A}$  by operators on a Hilbert space via the GNS construction. Since the algebra is generated by  $\varphi(f)$  the state is already determined by the  $n$ -point functions:

$$\omega_n(f_1, \dots, f_n) = \omega(\varphi(f_1) \dots \varphi(f_n)). \quad (2.11)$$

A special class of states is given by the *quasi free* states: All higher  $n$ -point functions are completely determined by the 2-point function  $\omega_2$ . Because of (2.10) the 2-point function has to fulfil the positivity condition:

$$\int dx dy \omega_2(x, y) \overline{f(x)} f(y) \geq 0. \quad (2.12)$$

## 3. The Fock space

From the notions above the construction of a representation space is just an application of the GNS theorem. We start by choosing a suitable 2-point function, which is given as usual by the positive frequency part of the commutator function. This completely defines a quasi free state on the algebra, called the vacuum state  $\omega_0$ .

We begin by introducing some abbreviations. The mass shell is defined by the hyperboloid

$$H_m = \{p \in \mathbb{R}^4, p^2 = m^2, p^0 > 0\}. \quad (2.13)$$

We denote a four vector on the mass shell by

$$H_m \ni \tilde{p} \doteq (E_{\mathbf{p}}, \mathbf{p}), \quad (2.14)$$

$$E_{\mathbf{p}} \doteq \sqrt{\mathbf{p}^2 + m^2}. \quad (2.15)$$

The invariant volume measure on the mass shell is given by

$$d\tilde{\mathbf{p}} \doteq \frac{d^3 \mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}. \quad (2.16)$$

With these abbreviations the commutator function is found to be

$$\Delta(x) = -2 \int d\tilde{\mathbf{p}} \sin(\tilde{p}x). \quad (2.17)$$

The verification of this expression only requires to check if it fulfils the Klein-Gordon equation (2.1) and the right initial values given by (2.3). The positive frequency part of  $\Delta$  is denoted by  $\Delta_+$ . It is given by:

$$\Delta_+(x) = \int d\tilde{\mathbf{p}} e^{-i\tilde{p}x}. \quad (2.18)$$

Corresponding to our state  $\omega_0$  we define the 2-point function by

$$\omega_0(\varphi(f)\varphi(g)) = i \langle f, \Delta_+ * g \rangle. \quad (2.19)$$

Bearing in mind that we are working with distributions we can write this as

$$\omega_2(x, y) = i\Delta_+(x - y). \quad (2.20)$$

Because of (2.8) and (2.9) the commutator is given by the asymmetric part of the 2-point function, hence:

$$i\Delta(x) = \Delta_+(x) - \Delta_+(-x). \quad (2.21)$$

Computing

$$\omega_2(\bar{f}, f) = \int dy dx \omega_2(x, y) \overline{f(x)} f(y) = \int d\tilde{\mathbf{p}} |\hat{f}(\tilde{p})|^2 \geq 0, \quad (2.22)$$

we find that the positivity condition (2.12) is fulfilled. Therefore  $\omega_2$  defines a positive semi definite scalar product on  $\mathcal{D}(\mathbb{M})$ . This generalizes to a positive semidefinite scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D}(\mathbb{M}^n)$  via

$$\langle f, g \rangle \doteq \int dx_1 \dots dx_n dy_1 \dots dy_n \prod_{i=1}^n \omega_2(x_i, y_i) \overline{f(x_1, \dots, x_n)} g(y_1, \dots, y_n). \quad (2.23)$$

Now consider the space of sequences of test functions  $(\Phi_n)_{n \geq 0}$  with:  $\Phi_0 \in \mathbb{C}$ ,  $\Phi_n \in \mathcal{D}(\mathbb{M}^n)$ , and  $\Phi_n$  is invariant under any permutation of its arguments. It is equipped with a positive semidefinite scalar product

$$\langle \Phi, \Psi \rangle \doteq \sum_{n=0}^{\infty} \langle \Phi_n, \Psi_n \rangle, \quad (2.24)$$

where the scalar product on the RHS is given by (2.23) for  $n > 0$  and  $\langle \Phi_0, \Psi_0 \rangle \doteq \Phi_0 \Psi_0$ . The space

$$\mathcal{F}(\mathcal{D}(\mathbb{R}^4)) \doteq \{ \Phi, \Phi_n \in \mathcal{D}(\mathbb{M}^n), \langle \Phi, \Phi \rangle < \infty \}, \quad (2.25)$$

is the symmetric Fock space . To define an action of  $\varphi$  on  $\mathcal{F}(\mathcal{D}(\mathbb{M}))$ , we have to define an action on a dense subspace, which consists of finite sequences only, denoted by

$$\mathcal{D} \doteq \{ \Phi \in \mathcal{F}, \exists m \in \mathbb{N}, \Phi_n = 0, \forall n > m \} \subset \mathcal{F}. \quad (2.26)$$

The field operator is decomposed into creation and annihilation operators according to  $\phi(f) = a(f) + a^*(f)$ . These operators act in Fock space in the following way:

$$(a(f)\Phi)_n(x_1, \dots, x_n) = \sqrt{n+1} \int dx dy f(x) \omega_2(x, y) \Phi_{n+1}(y, x_1, \dots, x_n) \quad (2.27)$$

$$(a^*(f)\Phi)_n(x_1, \dots, x_n) = \begin{cases} 0, & n = 0, \\ \frac{1}{\sqrt{n}} \sum_{k=1}^n f(x_k) \Phi_{n-1}(x_1, \dots, \hat{x}_k, \dots, x_n), & n \neq 0. \end{cases} \quad (2.28)$$

This automatically implements the commutation relations (2.8). Dividing out the ideal  $\mathcal{N}$  that is generated by the null space of the scalar product produces a pre Hilbert space  $\mathcal{H} = \mathcal{F}/\mathcal{N}$ . Because of (2.22)  $\mathcal{N}$  consists of all test functions  $f$  whose Fourier transform<sup>1</sup>  $\hat{f}(p_1, \dots, p_n)$  vanish if at least one momentum  $p_i$  is on the mass shell,  $p_i \in H_m$ .

The representative  $\Omega$  of the class  $(1, 0, 0, \dots) \in \mathcal{F}$  is called the vacuum vector. It defines a state  $\omega_0$  on  $\mathfrak{A}$  according to  $\omega_0(A) = (\Omega, A\Omega)$ . The scalar product  $(\cdot, \cdot)$  now is the positive definite one on the classes of  $\mathcal{H}$ . The state  $\omega_0$  has the two point function  $\omega_2$ .

The relations (2.27), (2.28) uniquely fix the higher  $n$ -point functions by  $\omega_2$  according to

$$\omega_{2n+1} = 0, \quad (2.29)$$

$$\omega_{2n}(x_1, \dots, x_{2n}) = \sum_{\text{pairings of } \{1, \dots, n\}} \prod_{\text{pairs } i < j} \omega_2(x_i, x_j). \quad (2.30)$$

Hence  $\omega_0$  is a quasi free state.

We define a unitary representation of the proper orthochronous Poincaré group on  $\mathcal{F}(\mathcal{D}(\mathbb{M}))$  by

$$(U(L)\Phi)_n(x_1, \dots, x_n) = \Phi_n(L^{-1}x_1, \dots, L^{-1}x_n), \quad \forall L \in \mathcal{P}_+^\uparrow, \quad (2.31)$$

with  $Lx = \Lambda x + a$ ,  $L^{-1}x = \Lambda^{-1}(x - a)$ ,  $L = (a, \Lambda)$ . That  $U$  is unitary follows easily from the fact that  $\omega_2$  is invariant:  $\omega_2(Lx, Ly) = \omega_2(x, y)$ . Then the field operator transforms according to

$$U(L)\varphi(x)U(L)^{-1} = \varphi(Lx). \quad (2.32)$$

#### 4. A remark on the commutator functions

Let us look how the different relatives of the commutator functions look like in our convention. With the support properties and our definition (2.6), (2.7) we have:

$$\Delta_{\text{ret}}(x) = \theta(x^0)\Delta(x), \quad (2.33)$$

$$\Delta_{\text{av}}(x) = -\theta(-x^0)\Delta(x) = \Delta_{\text{ret}}(-x). \quad (2.34)$$

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<sup>1</sup>All conventions and symbols are explained in appendix B

Inserting the representation (2.17) one finds:

$$\Delta_{\text{ret/av}}(x) = -\frac{1}{(2\pi)^4} \int d^4p \frac{e^{-ipx}}{p^2 - m^2 \pm i\epsilon p^0}. \quad (2.35)$$

Another very important distribution emerges from the time ordering of the 2-point function, namely the Feynman propagator.

$$i\Delta^F(x) \doteq \theta(x^0)\Delta_+(x) + \theta(-x^0)\Delta_+(-x) \quad (2.36)$$

$$= i\Delta_{\text{ret}}(x) - \Delta_+(-x) \quad (2.37)$$

$$= i\Delta_{\text{av}}(x) - \Delta_+(x) \quad (2.38)$$

$$= i\Delta^F(-x) \quad (2.39)$$

$$= -\frac{i}{(2\pi)^4} \int d^4p \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}. \quad (2.40)$$

Because of (2.7),  $\Delta^F$  is a Green's function, too:

$$(\square + m^2)\Delta^F = \delta. \quad (2.41)$$

An explicit configuration space expression of all these distributions can be found in [Sch95][chapter 2.3].





## CHAPTER 3

### Time ordered products

In the last chapter we have discussed the free scalar quantum field, obeying the Klein-Gordon equation. We introduce Wick polynomials of this field, that are composed operators and allow to define interactions, currents and the energy momentum tensor, for example. The introduction of a perturbative interaction into the theory requires the definition of time ordered products ( $T$ -products) of Wick polynomials. The naive ansatz for a time ordering prescription of  $n$  Wick monomials would be<sup>1</sup>

$$\begin{aligned} T(W_1(x_1) \dots W_n(x_n)) &= \\ &= \sum_{\pi \in S_n} \theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \dots \theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0) W_{\pi(1)}(x_{\pi(1)}) \dots W_{\pi(n)}(x_{\pi(n)}). \end{aligned} \tag{3.1}$$

Unfortunately the operators are distribution valued and the  $\theta$  function is not continuous at 0. Therefore the above products are not a priori well defined. As long as the  $W_i$  are linear in the fields (3.1) still works but already at the level of quadratic Wick monomials in the fields this naive ansatz breaks down leading to the well know ultra violet divergencies in generic (perturbative) quantum field theories. On the other hand we see that (3.1) gives a well defined expression as long as no points coincide.

In the definition of time ordered products the arguments are Wick products (or linear combinations of them). Hence different (looking) Wick monomials may be related through free field equations, e.g.  $\square\varphi$  and  $-m^2\varphi$  represent the same object. It turns out to be useful to solve this degeneration by introducing an abstract algebra  $\mathfrak{B}$  of auxiliary variables that is freely generated, as was shown by Boas [Boa99]. Then the time ordering becomes a map from  $\mathfrak{B}^n$  to operator valued distributions on  $\mathcal{D}$ . Hence, commutators and free field equations are formulated for first order  $T$ -products. Moreover this language is adapted to deal with couplings containing derivated fields, like Yang-Mills for example [Boa99]. We present the algebra  $\mathfrak{B}$  in section 1.

A very elegant solution for the definition of time ordering has been given by Epstein and Glaser [EG73], following ideas of Bogoliubov and Shirkov [BS76] and Stückelberg. A more accessible way was suggested by Stora [Sto93] and worked out in detail by Brunetti-Fredenhagen [BF96, BF00, Fre99]. We review their solution in section 2,3. Starting point is a set of axioms from which the causality – decoding the time ordering – is the most important one. Then,

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<sup>1</sup>If there are also fermions and ghost fields present, this ansatz requires a modification involving the sign of the permutation [Boa99].

for every order the products are determined up to the total diagonal (all points coincide) by the products of lower order. Therefore, one is left with an extension problem that can be solved in distribution theory.

As is well known in quantum field theory this solution is not unique in general. Therefore one forces the time ordering to respect certain symmetry relations. Moreover it has turned out to be useful to demand two more properties for the time ordered products: One deals with the case that one argument is a field (and imply the equations of motion for the interacting fields, defined in the next chapter). The other one relates the normalization of the operators to a normalization of vacuum expectation values [DF99]. All these conditions together are called normalization conditions. They are presented in section 4.

A very important normalization condition is given by the property of Poincaré covariance. Epstein and Glaser gave a proof of the existence of such a  $T$ -product for massive fields [EG73]. Later Stora-Popineau [SP82] and Dütsch et al. [DHKS94], [Sch95][chapter 4.5] found a cohomological existence proof that applies to arbitrary fields. In [BPP99] we have worked out their solution into an explicit form in lowest order perturbation theory. An inductive construction for higher orders was given in [Pra99b]. These preprints are subject of section 5.

### 1. The algebra of auxiliary variables

This section follows Boas [Boa99]. Since our work focuses on scalar fields we formulate this section for bosonic fields only and comment on the changes that are relevant if fermionic (or ghost) fields are present. The necessary modifications can be found in [Boa99].

The algebra  $\mathfrak{B}$  is defined as a freely generated  $*$ -algebra adapted to the fields that our quantum theory is formulated with. Assume our model contains  $r$  fields  $\varphi_1, \dots, \varphi_r$ . The  $\varphi_i$  are called *basic generators*. Additionally we have to consider spacetime derivatives, denoted by  $\varphi_{i,\mu_1\mu_2,\dots}$ . The elements of the set

$$\mathfrak{G} \doteq \{\varphi_i, \varphi_{i,\mu_1}, \varphi_{i,\mu_1\mu_2}, \dots, i = 1, \dots, r\} \quad (3.2)$$

are the *generators* of  $\mathfrak{B}$ . We define  $\mathfrak{B}$  as the unital free commutative algebra<sup>2</sup> generated by the elements of  $\mathfrak{G}$ . There is a natural definition of the derivation with respect to the generators according to<sup>3</sup>

$$\frac{\partial \varphi_i}{\partial \varphi_j} = \delta_i^j \mathbb{I}, \quad (3.3)$$

$$\frac{\partial}{\partial \varphi_j}(AB) = \frac{\partial A}{\partial \varphi_j} B + A \frac{\partial B}{\partial \varphi_j}, \quad \forall A, B \in \mathfrak{B}, \varphi_i, \varphi_j \in \mathfrak{G}. \quad (3.4)$$

Assume our free quantum field operators transform under the Lorentz group according to<sup>4</sup>

$$U(0, \Lambda) \varphi_i(x) U(0, \Lambda)^{-1} = D(\Lambda^{-1})_i^j \varphi_j(\Lambda x), \forall \Lambda \in \mathcal{L}_+^\uparrow, \quad (3.5)$$

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<sup>2</sup>Fermions (and ghosts) give rise to a charge and additional (anti-) commutators. The corresponding equivalence relation has to be divided out.

<sup>3</sup>The derivative becomes graded for fermions (and ghosts).

<sup>4</sup>We assume summation over double indices.

where  $D$  is a finite dimensional representation of  $\mathcal{L}_+^\dagger$  and  $U$  is a unitary representation of  $\mathcal{P}_+^\dagger$  on  $\mathcal{D}$ . The Lorentz group acts on  $\mathfrak{B}$  as an algebra homomorphism, i.e. a linear mapping which satisfies

$$D(\Lambda) \left( \prod_i \varphi_i \right) = \prod_i D(\Lambda)(\varphi_i), \quad \Lambda \in \mathcal{L}_+^\dagger, \varphi_i \in \mathfrak{G}. \quad (3.6)$$

Hence we only need to specify the action on the generators:

$$D(\Lambda)(\varphi_i) = D(\Lambda)_i^j \varphi_j, \quad (3.7)$$

$$D(\Lambda)(\varphi_{i,\mu_1 \dots \mu_n}) = \Lambda_{\mu_1}^{\nu_1} \dots \Lambda_{\mu_n}^{\nu_n} D(\Lambda)_i^j \varphi_{j,\nu_1 \dots \nu_n}. \quad (3.8)$$

At the end we have additionally a  $*$ -involution acting on  $\mathfrak{B}$  according to

$$(aAB)^* = \bar{a}B^*A^*. \quad (3.9)$$

For one scalar field this is obviously trivial

$$\varphi^* = \varphi, \quad \text{and} \quad \varphi_{,\nu_1 \dots \nu_n}^* = \varphi_{,\nu_1 \dots \nu_n}. \quad (3.10)$$

Let us mention that the goal of this algebra is that a field and its derivatives are treated as independent objects, thus allowing to uniquely define a derivative with respect to a generator. This symbolic derivation is the same one uses in classical mechanics for deriving the Euler-Lagrange equations, for example.

Now we discuss the mapping of elements of  $\mathfrak{B}$  to Wick polynomials on  $\mathcal{D}$ . This is obtained by the time ordering map  $T$  of one argument:

$$T : \mathfrak{B} \mapsto \text{Dist}_1(\mathcal{D}), \quad (3.11)$$

where  $\text{Dist}_1$  denotes the space of operator valued distributions on  $\mathcal{D}$ , namely the set of linear continuous maps  $\mathcal{D}(\mathbb{M}) \mapsto \text{End}(\mathcal{D})$ . The mapping  $T$  is  $\mathbb{C}$ -linear. But it is no algebra homomorphism since there exists no product because of the distributional character of the images. On the generators the map is defined as follows:

$$T(\varphi_i)(x) \doteq \varphi_i(x) \quad (3.12)$$

$$T(\varphi_{i,\nu_1 \dots \nu_n})(x) \doteq \partial_{\nu_1} \dots \partial_{\nu_n} \varphi_i(x). \quad (3.13)$$

The free fields are quantized according to

$$[T(\varphi_i)(x), T(\varphi_j)(y)] = i\Delta_{ij}(x-y), \quad \forall \varphi_i, \varphi_j \in \mathfrak{G} \subset \mathfrak{B}. \quad (3.14)$$

Since the indices  $i, j$  also represent higher generators the corresponding commutator contributions are given by

$$\Delta_{i,\mu_1 \dots \mu_n \ j,\nu_1 \dots \nu_m} = (-)^m \partial_{\mu_1} \dots \partial_{\mu_n} \partial_{\nu_1} \dots \partial_{\nu_m} \Delta_{ij}. \quad (3.15)$$

The equations of motion read:

$$D_{ij}T(\varphi_j) = 0, \quad (3.16)$$

where  $D$  is a hyperbolic partial differential operator. Then we define the mapping of general elements of  $W \in \mathfrak{B}$  through the implicit formula:

$$[T(W)(x), T(\varphi_i)(y)] = iT \left( \frac{\partial W}{\partial \varphi_j} \right) (x) \Delta_{ji}(x-y), \quad (3.17)$$

$$\omega_0(T(W)(x)) = 0. \quad (3.18)$$

As was shown by Boas this fixes the  $T(W)$  uniquely: Let  $W = \prod_i \varphi_i$  be a monomial, then  $T(W) =: \prod_i \varphi_i :$ . The colons denote *Wick* (or *normal*) *ordering* which is defined by the recursion

$$:\mathbb{I}: = \mathbb{I}, \quad (3.19)$$

$$:\varphi_i(x): = \varphi_i(x), \quad (3.20)$$

$$\begin{aligned} :\varphi_{i_1}(x_1) \dots \varphi_{i_n}(x_n): &= :\varphi_{i_1}(x_1) \dots \varphi_{i_{n-1}}(x_{n-1}) : \varphi_{i_n}(x_n) + \\ &\quad - \sum_{i=1}^{n-1} \omega_{2i, i_n}(x_i, x_n) :\varphi_{i_1}(x_1) \dots \not{\varphi}_{i_n}(x_n) :. \end{aligned} \quad (3.21)$$

Since (2.30) is just the summation of this recursion we have

$$\omega_0(:\varphi_{i_1}(x_1) \dots \varphi_{i_n}(x_n):) = 0. \quad (3.22)$$

The normal ordering allows to restrict the distributional operators to any sub manifold of coinciding points. This provides us with composed fields like

$$:\varphi(x)^2:, :\varphi(x)^{18} \square \varphi(x):, :\partial_\mu \varphi(x) \partial_\nu \varphi(x):, \dots \quad (3.23)$$

From the definition (3.19)–(3.21) follows, that derivations commute with the Wick ordering and the free field equations hold inside the Wick colons.

Because of the free field equations the map  $T$  defines a non faithful representation of  $\mathfrak{B}$  in  $\text{Dist}_1(\mathcal{D})$ .

## 2. The axioms for time ordered products

We have related the symbolic algebra  $\mathfrak{B}$  to the vector space of Wick polynomials by the  $T$  operation of one argument. Now we formulate the axioms that make  $T$  a time ordered product of  $n$  arguments that reduces to the naive ansatz (3.1) for non coincident points.

Let us denote elements of  $\mathfrak{B}$  by  $W_i$  such that under the map  $T : W_i \mapsto T(W_i)$  represents the Wick polynomials whose time ordering should be defined. The space of distributions on  $\mathcal{D}(\mathbb{M}^n)$  with values in  $\text{End}(\mathcal{D})$  is denoted by  $\text{Dist}_n(\mathcal{D})$ . We require the T-products to fulfil the following axioms:

**P1. Well-posedness.** The time ordered products of  $n$  symbols denoted by  $T(W_1, \dots, W_n)(x_1, \dots, x_n)$  are multi linear<sup>5</sup> strongly continuous maps  $\mathfrak{B}^n \mapsto \text{Dist}_n(\mathcal{D})$ .

**P2. Symmetry.** The time ordered products are invariant under any permutation of their arguments,<sup>6</sup>

$$T(W_{\pi(1)}, \dots, W_{\pi(n)})(x_{\pi(1)}, \dots, x_{\pi(n)}) = T(W_1, \dots, W_n)(x_1, \dots, x_n). \quad (3.24)$$

$$\forall \pi \in S_n.$$

<sup>5</sup>We also allow for the mapping  $(\mathcal{C}^\infty \otimes \mathfrak{B})^n \mapsto \text{Dist}_n(\mathcal{D})$  according to  $T(f_1 W_1, \dots, f_n W_n)(x_1, \dots, x_n) = f(x_1) \dots f(x_n) T(W_1, \dots, W_n)(x_1, \dots, x_n)$ , which for  $f_i = 1, \forall i$ , reduces to the above case.

<sup>6</sup>If fermions or ghost are present there is an additional (-1)-factor corresponding to the sign of the permutation, see [Boa99].

**P3. Causality.** Assume the points  $x_1, \dots, x_n$  can be separated by a space like hypersurface, such that  $x_i \gtrsim x_j, \forall i = 1, \dots, k, j = k+1, \dots, n$ , with the notation:  $x \gtrsim y \Leftrightarrow y \notin \overline{V}_+(x)$ . Then the  $T$ -products factorize:

$$\begin{aligned} T(W_1, \dots, W_n)(x_1, \dots, x_n) &= \\ &= T(W_1, \dots, W_k)(x_1, \dots, x_k) T(W_{k+1}, \dots, W_n)(x_{k+1}, \dots, x_n). \end{aligned} \quad (3.25)$$

**P4. Translation covariance.** Under a translation the  $T$ -product transforms according to

$$(\text{Ad}U(\mathbb{I}, a))T(W_1, \dots, W_n)(x_1, \dots, x_n) = T(W_1, \dots, W_n)(x_1 + a, \dots, x_n + a). \quad (3.26)$$

We use the abbreviation  $T(I)(x_I) \doteq T(W_i, i \in I)(x_i, i \in I)$ . The causality condition **P3** implies that space like separated time ordered products commute, since  $I \sim J \Leftrightarrow (I \gtrsim J) \wedge (I \lesssim J)$  implies

$$T(I \cup J)(x_{I \cup J}) = T(I)(x_I)T(J)(x_J) = T(J)(x_J)T(I)(x_I). \quad (3.27)$$

We have demanded multi linearity in **P1**. Therefore it is important that the arguments of the  $T$ -products are from our algebra  $\mathfrak{B}$  where no equations of motion hold. Otherwise all time ordered products containing the Wick polynomial  $:\varphi \square \varphi: + :m^2 \varphi^2:$  would be zero for example. Although this choice provides for a well defined time ordering prescription we encounter a situation where one needs a non zero definition, i.e. in the energy momentum tensor.

### 3. The inductive construction

The last section has stated the axioms for time ordered products. Now we formulate an inductive construction: under the assumption that all products are known up to order  $n-1$  we show how to derive  $T$  in order  $n$ . This procedure goes back to Epstein and Glaser [EG73]. They determined a causal distribution through  $T$ -products of lower order and developed a procedure for a causal splitting into a retarded and advanced supported part – analogous to the decomposition of  $\Delta$  into  $\Delta_{\text{ret}}$  and  $\Delta_{\text{av}}$  but also applicable to more singular distributions. Then the time ordered product can be expressed by the splitting solution.

Later, Stora has suggested a method to derive the time ordered products directly, without the detour using the advanced and retarded distributions [Sto93]. Brunetti-Fredenhagen [BF96, BF00] have provided a complete analysis (and moreover generalized the whole construction for the treatment of scalar fields on curved spacetime) based on that idea which we review here.

**3.1. The induction start.** If we have no arguments for our  $T$ -product we set

$$T(\emptyset) = \mathbb{I} \in \text{End}(\mathcal{D}). \quad (3.28)$$

The  $T$ -products of one argument were already defined in the last section according to

$$T(W)(x) = :W(x): \in \text{Dist}_1(\mathcal{D}). \quad (3.29)$$

We proceed with the

**3.2. Recursion to higher orders.** Assume that all time ordered products have been constructed and satisfy the axioms **P1**–**P4**. Let us use the abbreviation  $N \doteq \{1, \dots, n\}$ . Then we define the sets:

$$\mathcal{C}_I \doteq \{(x_1, \dots, x_n) \in \mathbb{M}^n | x_i \notin \overline{V}_-(x_j), \forall i \in I, j \in I^c\}, \quad (3.30)$$

and  $I^c \doteq N \setminus I$  is the complement in  $N$ . Let  $\text{Diag}_n \doteq \{(x_1, \dots, x_n) \in \mathbb{M}^n | x_1 = \dots = x_n\}$  be the diagonal in  $\mathbb{M}^n$ , then

$$\bigcup_{\substack{I \subset N \\ I \neq \emptyset \\ I \neq N}} = \mathbb{M}^n \setminus \text{Diag}_n. \quad (3.31)$$

Then, on any  $\mathcal{C}_I$  we set

$$T_I(N)(x_N) \doteq T(I)(x_I)T(I^c)(x_{I^c}). \quad (3.32)$$

Now one has to glue together all  $T_I$  to a distribution on  $\mathbb{M}^n \setminus \text{Diag}_n$ . But since different  $\mathcal{C}_I$  can overlap one has to check, that the compatibility condition<sup>7</sup>

$$T_I \upharpoonright_{\mathcal{C}_I \cap \mathcal{C}_J} = T_J \upharpoonright_{\mathcal{C}_I \cap \mathcal{C}_J}, \quad (3.33)$$

holds, if  $\mathcal{C}_I \cap \mathcal{C}_J \neq \emptyset$ . But this follows from causality (**P3**) in lower orders: Since  $J \gtrsim J^c$  and  $I \gtrsim I^c$  we have

$$T_I(N) = T(I)T(I^c) \quad (3.34)$$

$$= T(I \cap J)T(I \cap J^c)T(I^c \cap J)T(I^c \cap J^c) \quad (3.35)$$

$$= T(I \cap J)T(I^c \cap J)T(I \cap J^c)T(I^c \cap J^c) \quad (3.36)$$

$$= T(J)T(J^c) \quad (3.37)$$

$$= T_J(N), \quad (3.38)$$

where the two inner  $T$ -products in (3.36) commute since  $I^c \cap J \sim I \cap J^c$ . Now we take a locally finite smooth partition of unity  $\{f_I\}_{I \in N, I \neq \emptyset, I \neq N}$  on  $\mathbb{M}^n \setminus \text{Diag}_n$  subordinate to  $\{\mathcal{C}_I\}_{I \in N, I \neq \emptyset, I \neq N}$ :

$$\sum_{\substack{I \subset N \\ I \neq \emptyset \\ I \neq N}} f_I = 1 \text{ on } \mathbb{M}^n \setminus \text{Diag}_n, \text{ supp } f_I \subset \mathcal{C}_I. \quad (3.39)$$

Then we define:

$${}^0T(N) \doteq \sum_{\substack{I \subset N \\ I \neq \emptyset \\ I \neq N}} f_I T_I(N). \quad (3.40)$$

It can be verified that this definition does not depend on the choice of the partition of unity  $\{f_I\}$  and that  ${}^0T$  is a well defined operator valued distribution on  $\mathcal{D}(\mathbb{M}^n)$  satisfying **P1** – **P4**.

By construction  ${}^0T$  is a linear combination of numerical translation invariant distributions multiplied with certain Wick products:

$${}^0T(N)(x_N) = {}^0t(x_1 - x_2, \dots, x_{n-1} - x_n) : V_1(x_1) \dots V_n(x_n) : . \quad (3.41)$$

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<sup>7</sup>In the following we frequently omit the arguments  $x_I$  since they are already determined by the sets  $I$  in  $T(I)$ .

As was shown by Epstein-Glaser, these products always exist [EG73][Theorem 0]. Hence the definition of  $T$  reduces to finding an extension of the numerical distribution  ${}^0t$  to  $\text{Diag}_n$ , which in difference coordinates translates into the problem of extending a distribution to the origin. This problem is addressed in the next subsection.

**3.3. Extension of distributions to the origin.** The solution of this problem requires the introduction of a quantity that measures the singularity of the distribution at the origin [Ste71].

DEFINITION 1. A distribution  $t \in \mathcal{D}'(\mathbb{R}^d)$  has scaling degree  $s$  at  $x = 0$ , if

$$s = \inf\{s' \in \mathbb{R} \mid \lambda^{s'} T(\lambda x) \xrightarrow{\lambda \searrow 0} 0 \text{ in the sense of distributions}\}. \quad (3.42)$$

We set  $\text{scal deg}(t) \doteq s$  and define  $\text{sing ord}(t) := [s] - n$ , the singular order.<sup>8</sup>

The definition also holds if  $t \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ . One easily finds that differentiation increases the scaling degree while multiplication with  $x$  decreases it:

$$\text{scal deg}(x^\beta t) = \text{scal deg}(t) - |\beta|, \quad (3.43)$$

$$\text{scal deg}(\partial^\beta t) = \text{scal deg}(t) + |\beta|, \quad (3.44)$$

$$\text{scal deg}(ft) \leq \text{scal deg}(t), \quad (3.45)$$

for all  $t \in \mathcal{D}'(\mathbb{R}^d)$  and  $f \in \mathcal{D}(\mathbb{R}^d)$ . Now, the solution of the extension problem depends on the sign of the singular order.

THEOREM 1. Let  ${}^0t \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$  with scaling degree  $s < n$ . Then there exists a unique  $t \in \mathcal{D}'(\mathbb{R}^d)$  with scaling degree  $s$  and  $t(f) = {}^0t(f)$  for all  $f \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ .

Otherwise we introduce the

$W$ -OPERATION. Let  $\mathcal{D}^\omega(\mathbb{R}^d)$  be the subspace of test functions vanishing up to order  $\omega$  at 0. Define

$$\begin{aligned} W_{(\omega;w)} : \mathcal{D}(\mathbb{R}^d) &\mapsto \mathcal{D}^\omega(\mathbb{R}^d), \quad f \mapsto W_{(\omega;w)} f, \\ (W_{(\omega;w)} f)(x) &= f(x) - w(x) \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \left( \partial^\alpha \frac{f}{w} \right) (0), \end{aligned} \quad (3.46)$$

with  $w \in \mathcal{D}(\mathbb{R}^d)$ ,  $w(0) \neq 0$ .

Now we can discuss the general case.

THEOREM 2. Let  ${}^0t \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$  with scaling degree  $s \geq n$ . Given  $w \in \mathcal{D}(\mathbb{R}^d)$  with  $w(0) \neq 0$  and constants  $c^\alpha \in \mathbb{C}$  for all multi indices  $\alpha$ ,  $|\alpha| \leq \omega$ , then there is exactly one distribution  $t' \in \mathcal{D}'(\mathbb{R}^d)$  with scaling degree  $s$  and following properties:

- I.  $\langle t', f \rangle = \langle {}^0t, f \rangle \quad \forall f \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ ,
- II.  $\langle t', wx^\alpha \rangle = c^\alpha$ ,

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<sup>8</sup> $[s]$  is the largest integer that is smaller than or equal to  $s$ .

with  $t'$  given by:

$$\langle t', f \rangle = \langle t, W_{(\omega;w)} f \rangle + \sum_{|\alpha| \leq \omega} \frac{c^\alpha}{\alpha!} \left( \partial^\alpha \frac{f}{w} \right) (0). \quad (3.47)$$

Here  $t$  is the unique extension by theorem 1,  $W_{(\omega;w)}$  is given by (3.46) and  $\omega$  is the singular order of  ${}^0t$ .

With these theorems the extension can be done right away. In the case of non negative singular order we notice that an ambiguity appears, namely all choices of the constants  $c^\alpha$  yield a well defined solution. The normalization condition in the next section restricts this freedom further. Before we proceed to these conditions we introduce the

**3.4. Anti time ordered products.** If one wants to define interacting perturbative fields one could equally well have started with the definition of anti chronological products, that give a meaning to the expression

$$\begin{aligned} \bar{T}(W_1, \dots, W_n)(x_1, \dots, x_n) &= \\ &= \sum_{\pi \in S_n} \theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \dots \theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0) W_{\pi(n)}(x_{\pi(n)}) \dots W_{\pi(1)}(x_{\pi(1)}) \end{aligned} \quad (3.48)$$

in the case of coinciding points, corresponding to equation (3.1). The causality property that encodes the wanted anti chronological factorization reads according to **P3**:

$$\bar{T}(N)(x_N) = \bar{T}(I^c)(x_{I^c}) \bar{T}(I)(x_I), \text{ if } I \gtrsim I^c. \quad (3.49)$$

It turns out in the next chapter that the functional of the anti chronological products is the inverse  $S$ -matrix hence we have

$$\sum_{I \in N} (-)^{|I|} \bar{T}(I)(x_I) T(I^c)(x_{I^c}) = \sum_{I \in N} (-)^{|I^c|} T(I)(x_I) \bar{T}(I^c)(x_{I^c}) = 0. \quad (3.50)$$

This equation allows a recursive definition of the  $\bar{T}$ -products of order  $n$  by lower order  $\bar{T}$ - and all order  $T$ -products:

$$\bar{T}(N)(y_N) = - \sum_{\substack{I \subset N \\ I \neq \emptyset}} (-)^{|I|} T(I)(x_I) \bar{T}(I^c)(x_{I^c}), \quad (3.51)$$

$$= - \sum_{I \subsetneq N} (-)^{|I^c|} \bar{T}(I)(x_I) T(I^c)(x_{I^c}). \quad (3.52)$$

Explicit solution of the recursion gives:

$$\bar{T}(N)(x_N) = \sum_{P \in \text{Part}(N)} (-)^{|P|+|N|} \prod_{Q \in P} T(Q)(x_Q). \quad (3.53)$$



#### 4. Normalization conditions

As was seen in the last section the extension procedure that has to be applied to the numerical distribution  ${}^0t$  in order to define the  $T$ -products everywhere produces an ambiguity related to the constants  $c^\alpha$  from theorem 2. We formulate a set of conditions, that apply to these constants in a way that certain properties for the complete  $T$ -products are fulfilled. These conditions were introduced in [DF99] and afterwards generalized to the case, when derivated fields are present in [Boa99].

The first normalization condition deals with the possible occurrence of discrete symmetries of  ${}^0T$ . Assume, there is a finite group  $G$  acting invariantly on  ${}^0T$  according to  $: {}^0T \mapsto {}^0T_a$ , and  ${}^0T_a = {}^0T_{\mathbb{I}}$  for all  $a \in G$ . The extension of  ${}^0T_a$  is denoted by  $T_a$ . We require the extension to be invariant under  $G$ , too:

$$T_a(W_1, \dots, W_n)(x_1, \dots, x_n) = T_{\mathbb{I}}(W_1, \dots, W_n)(x_1, \dots, x_n), \quad \forall W_i \in \mathfrak{B}, a \in G. \quad (\mathbf{N0})$$

Obviously this condition can always be fulfilled by choosing

$$T(W_1, \dots, W_n)(x_1, \dots, x_n) = \frac{1}{|G|} \sum_{a \in G} T_a(W_1, \dots, W_n)(x_1, \dots, x_n) \quad (3.54)$$

This condition already establishes the correct  $P, C, T$ -transformation properties of the  $T$ -products, see ([Sch95]). Since this condition is quite trivial, we only talk of a  $T$ -product if it is fulfilled.

The next condition implements the conservation of Poincaré covariance. Let  $\mathcal{P}_+^\uparrow$  act on  $\text{End}(\mathcal{D})$  through the representation  $U$ . Then we demand:

$$\begin{aligned} (\text{Ad}U(L))T(W_1, \dots, W_n)(x_1, \dots, x_n) &= \\ &= T(D(\Lambda^{-1})(W_1), \dots, D(\Lambda^{-1})(W_n))(Lx_1, \dots, Lx_n), \quad (\mathbf{N1}) \end{aligned}$$

for all  $L = (a, \Lambda) \in \mathcal{P}_+^\uparrow$  and the action of the representation  $D$  on  $\mathfrak{B}$  is defined in (3.7), (3.8). Let us remark that our current formulation is redundant since **N3** already contains the axiom **P4** namely for  $\Lambda = \mathbb{I}$ . Nevertheless we stick to this formulation for the following reason: It was already remarked by Epstein-Glaser [EG73] that translation covariance is a crucial condition for the causal construction. In that case their theorem 0 always guarantees a solution in the form (3.41). In contrast to **P4** normalization condition **N1** is not necessary for performing the inductive construction.

Moreover Brunetti-Fredenhagen [BF96, BF00] have shown that the inductive construction can also be performed if **P4** is replaced by a weaker assumption formulated with the techniques of micro local analysis. Their approach also applies for curved spacetime. Since there is no symmetry in a generic spacetime a normalization condition like **N1** reflecting the symmetry of the Minkowski space has to be abandoned.

It was shown by Epstein and Glaser [EG73] that **N1** can be fulfilled. In the next section we derive a procedure for an explicit construction.

The anti time ordered products were introduced in the last chapter. Then unitarity is fulfilled if

$$T(W_1, \dots, W_n)(x_1, \dots, x_n)^* = \overline{T}(W_n^*, \dots, W_1^*)(x_n, \dots, x_1), \quad \forall W_i \in \mathfrak{B}. \quad (\mathbf{N2})$$

The  $*$  on the LHS is the adjoint on  $\mathcal{D}$ . On the RHS the order of the symbols is reversed. It can be rearranged using **P3**. Epstein and Glaser have shown that **N2** can always be satisfied: Assume it holds to order  $n-1$  together with **N1**. For every normalization  $T' = T(W_1, \dots, W_n)$  that fulfils **N1** we can form  $T = \frac{1}{2}(T' + \overline{T'}^*)$  which satisfies **N2**. Then **N1** is fulfilled automatically since  $U$  is unitary:  $U^* = U^{-1}$ .

If we commute the  $T$ -product with a generator we require (compare to (3.17)):

$$\begin{aligned} [T(W_1, \dots, W_n)(x_1, \dots, x_n), \varphi_i(y)] = \\ = i \sum_{k=1}^n T \left( W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) (x_1, \dots, x_n) \Delta_{ji}(x_k - y), \end{aligned} \quad (\mathbf{N3})$$

for every  $W_i \in \mathfrak{B}$  and  $\varphi_i(y) = T(\varphi_i)(y)$ ,  $\varphi_i \in \mathfrak{G}$ . Since the center of  $\text{End}(\mathcal{D})$  only contains multiples of the identity [Sch95], **N3** determines the  $T$ -product up to a  $\mathbb{C}$ -number distribution by the  $T$ -products of the sub monomials. This distribution is given by the vacuum expectation value. It was shown by Boas that **N3** can always be satisfied (together with **N1** and **N2**) and is equivalent to the *causal Wick expansion*:

$$\begin{aligned} T(W_1, \dots, W_n)(x_1, \dots, x_n) = \\ = \sum_{\gamma_1, \dots, \gamma_n} \omega_0 \left( T \left( W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)} \right) (x_1, \dots, x_n) \right) \frac{:\varphi^{\gamma_1}(x_1) \cdots \varphi^{\gamma_n}(x_n):}{\gamma_1! \cdots \gamma_n!} \end{aligned} \quad (3.55)$$

Here the  $\gamma_i \in \mathbb{N}^r$  are multi indices with one entry for each of the  $r$  generators in  $\mathfrak{G}$ , i.e.

$$\gamma_i = ((\gamma_i)_1, \dots, (\gamma_i)_r) \in \mathbb{N}^r \quad (3.56)$$

The  $W^{(\gamma_i)}$  are derivatives with respect to the generators

$$W^{(\gamma_i)} \doteq \frac{\partial^{|\gamma_i|} W}{\partial^{(\gamma_i)_1} \varphi_1 \cdots \partial^{(\gamma_i)_r} \varphi_r}, \quad (3.57)$$

where  $|\gamma_i| = \sum_{k=1}^r (\gamma_i)_k$ . The  $\varphi^{\gamma_i}$  are defined by

$$\varphi^{\gamma_i}(x) \doteq T \left( \prod_{k=1}^r \varphi_k^{(\gamma_i)_k} \right) (x), \quad (3.58)$$

and

$$(\gamma_i)! \doteq \prod_{k=1}^r (\gamma_i)_k!. \quad (3.59)$$

The following normalization condition is a partial differential equation concerning  $T$ -products with only one generator.

$$\begin{aligned} D_{ij}^x T(W_1, \dots, W_n, \varphi_j)(x_1, \dots, x_n, z) = \\ = i \sum_{k=1}^n T \left( W_1 \dots \frac{\partial W_k}{\partial \varphi_i} \dots W_n \right) (x_1, \dots, x_n) \delta(x_k - z), \quad (\mathbf{N4}) \end{aligned}$$

where  $W_i \in \mathfrak{B}$  and  $\varphi_i \in \mathfrak{G}$ . This differential equation can be solved by

$$\begin{aligned} T(W_1, \dots, W_n \varphi_i)(x_1, \dots, x_n, z) = \\ = i \sum_{k=1}^n \Delta_{ij}^F(z - x_k) T \left( W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) (x_1, \dots, x_n) + \\ + \sum_{\gamma_1 \dots \gamma_n} \omega_0 \left( T \left( W_1^{(\gamma_1)}, \dots, W_n^{(\gamma_n)} \right) (x_1, \dots, x_n) \right) \frac{:\varphi^{\gamma_1}(x_1) \dots \varphi^{\gamma_n}(x_n) \varphi_i(z):}{\gamma_1! \dots \gamma_n!}. \end{aligned} \quad (3.60)$$

All proofs of the compatibility statements can be found in [Boa99]. Moreover it is shown that the normalization conditions also hold in the presence of derivated fields in the  $W_i$ . This can be accomplished by a suitable adaption of the differential operator  $D_{ij}$ , the commutator functions  $\Delta_{ij}$  and the Feynman propagators  $\Delta_{ij}^F(x - y) = \omega_0(T(\varphi_i, \varphi_j)(x, y))$ .

As we have seen in the last subsection the definition of  $T$  requires an extension of the numerical distributions of the decomposition (3.55). We do not want to increase the scaling degree of these distribution in the renormalization process. Then the scaling degree of the time ordered numerical distributions is already determined by the dimensions of the symbols from  $\mathfrak{B}$ :

$$\text{scal deg } \omega_0(T(W_1, \dots, W_n)) \Big|_{\text{Diag}_n} = \sum_{i=1}^n \dim W_i, \quad (3.61)$$

for all monomials  $W_i \in \mathfrak{B}$ . Since in  $\mathfrak{B}$  every  $W$  can be decomposed uniquely into basic generators and derivatives we have according to this decomposition:

$$\dim W = \sum_{i \in \mathfrak{G}_b} \dim \varphi_i + \#\partial, \quad (3.62)$$

and the dimension of the bosonic (fermionic) fields is one (3/2).

## 5. Poincaré covariance

In the last section we have demanded Poincaré covariance to hold for the  $T$ -products by condition **N2**. In the decomposition of the products according to the causal Wick expansion (3.55) we see that covariance properties have to be fulfilled by the numerical distributions only, since the Wick products already transform correctly under Poincaré transformations. Since we are working in an inductive procedure we require covariance to be fulfilled in lower orders and have to provide an extension of the numerical distributions according to theorem 2 that respects these properties. Therefore the solution of this problem reduces to finding a suitable set of constants  $c^\alpha$ .

Epstein-Glaser [EG73] gave an existence proof for the *central solution* of the subtraction procedure in case of a massive theory. It corresponds to the choice  $w = 1$  and  $c^\alpha = 0$  in theorem 2 which obviously preserves Lorentz covariance. For the massless case they suggested an averaging over a maximal compact subgroup of the complexified Lorentz group.

Another proof was given by Stora and Popineau [SP82][unpublished] and Dütsch et al. [DHKS94]. A detailed representation can be found in the book of Scharf [Sch95]. It is based on cohomological arguments. We review their analysis here, adapted to our case.

In [BPP99] we have explicitly calculated the constants in lowest order perturbation theory for scalar fields. This solution may also apply in the case of special symmetry in higher orders [Pin99]. A general solution for higher orders and arbitrary covariance could be derived only recursively in [Pra99b]. This section reviews the content of these preprints.

**5.1. The subtraction procedure.** We remind the reader of the subtraction operator  $W$  (3.46) which was necessary to define the distributional extension. The  $W$ -operation is simplified if we require  $w(0) = 1$  and  $\partial^\alpha w(0) = 0$ , for  $0 < |\alpha| \leq \omega$  (this was our assumption in [BPP99]). A test function with these properties can be derived from an arbitrary test function by application of the following  $V$ -operation ( $\partial^\mu w^{-1}$  means  $\partial^\mu(w^{-1})$ ):

$$V_\omega : \mathcal{D}(\mathbb{R}^d) \mapsto \mathcal{D}(\mathbb{R}^d), \quad (V_\omega w)(x) \doteq w(x) \sum_{|\mu| \leq \omega} \frac{x^\mu}{\mu!} \partial^\mu w^{-1}(0), \quad (3.63)$$

where  $w(0) \neq 0$  is still assumed. We can write  $W$  as

$$(W_{(\omega;w)} f)(x) = f(x) - \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} V_{\omega-|\alpha|} w \partial^\alpha f(0). \quad (3.64)$$

Let us denote the extension corresponding to this subtraction by  $t_{(\omega;w)}$ :

$$\langle t_{(\omega;w)}, f \rangle \doteq \langle {}^0 t, W_{(\omega;w)} f \rangle. \quad (3.65)$$

Adding any polynomial in derivatives of  $\delta$  up to order  $\omega$  produces another extension  $\bar{t}_{(\omega;w)}$ :

$$\langle \bar{t}_{(\omega;w)}, f \rangle = \langle t_{(\omega;w)}, f \rangle + \sum_{|\alpha| \leq \omega} \frac{a^\alpha}{\alpha!} \partial^\alpha f(0), \quad (3.66)$$

or rearranging the coefficients

$$= \langle t_{(\omega;w)}, f \rangle + \sum_{|\alpha| \leq \omega} \frac{c^\alpha}{\alpha!} \partial^\alpha (f w^{-1})(0) \quad (3.67)$$

Since  $W_{(\omega;w)}(w x^\alpha) = W_{(\omega;w)}(x^\alpha V_{\omega-|\alpha|} w) = 0$  for  $|\alpha| \leq \omega$ ,  $c$  resp.  $a$  are given by

$$a^\alpha = \langle \bar{t}_{(\omega;w)}, x^\alpha V_{\omega-|\alpha|} w \rangle, \quad c^\alpha = \langle \bar{t}_{(\omega;w)}, x^\alpha w \rangle. \quad (3.68)$$

They are related through:

$$\begin{aligned} a^\alpha &= c^\alpha \sum_{|\mu| \leq \omega - |\alpha|} \frac{c^\mu}{\mu!} \partial_\mu w^{-1}(0), & c^\alpha &= a^\alpha \sum_{|\mu| \leq \omega - |\alpha|} \frac{a^\mu}{\mu!} \partial_\mu w(0), & 1 \leq |\alpha| \leq \omega, \\ a^0 &= \sum_{|\mu| \leq \omega} \frac{c^\mu}{\mu!} \partial_\mu w^{-1}(0), & c^0 &= \sum_{|\mu| \leq \omega} \frac{a^\mu}{\mu!} \partial_\mu w(0). \end{aligned} \quad (3.69)$$

The equation for  $a$  follows from the Leibnitz rule in (3.67), while the equation for  $c$  is derived from (3.68).

**5.2. The  $G$ -covariant extension.** We begin defining the notion of a  $G$ -covariant distribution. So let  $G$  be a linear transformation group on  $\mathbb{R}^d$  i.e.  $x \mapsto gx, g \in G$ . Then

$$x^\alpha \mapsto g^\alpha_\beta x^\beta = (gx)^\alpha \quad (3.70)$$

denotes the corresponding tensor representation.  $G$  acts on functions in the following way:

$$(gf)(x) \doteq f(g^{-1}x), \quad (3.71)$$

so that  $\mathcal{D}$  is made a  $G$ -module. We further have

$$g(fh) = (gf)(gh), \quad (3.72)$$

$$x^\alpha \partial_\alpha (g^{-1}f) = (gx)^\alpha g^{-1}(\partial_\alpha f), \quad (3.73)$$

$$x^\alpha \partial_\alpha (g^{-1}f)(0) = (gx)^\alpha \partial_\alpha f(0) \forall g \in G, f, h \in \mathcal{D}(\mathbb{R}^d) \quad (3.74)$$

Now assume we have a distribution  ${}^0t \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\}, V)$  taking values in a finite vector space  $V$  that serves as a representation space for the group  $G$ . The distribution transforms covariantly under the Group  $G$  as a density, i.e.

$${}^0t(gx)|\det g| = D(g){}^0t(x), \quad (3.75)$$

where  $D$  is the corresponding representation. That means:

$$\langle {}^0t, gf \rangle = \langle D(g){}^0t, f \rangle \doteq D(g) \langle {}^0t, f \rangle. \quad (3.76)$$

We now investigate the covariance properties in the extension process. We compute:

$$\begin{aligned}
D(g) \langle t_{(\omega;w)}, g^{-1}f \rangle - \langle t_{(\omega;w)}, f \rangle &= \\
&= D(g) \langle {}^0t, W_{(\omega;w)} g^{-1}f \rangle - \langle {}^0t, W_{(\omega;w)} f \rangle \\
&\stackrel{(3.64)}{=} D(g) \left\langle {}^0t, g^{-1}f - \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} V_{\omega-|\alpha|} w \partial_\alpha (g^{-1}f)(0) \right\rangle - \langle {}^0t, W_{(\omega;w)} f \rangle \\
&\stackrel{(3.72, 3.74)}{=} D(g) \left\langle {}^0t, g^{-1} \left( f - \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} (g V_{\omega-|\alpha|} w) \partial_\alpha f(0) \right) \right\rangle - \langle {}^0t, W_{(\omega;w)} f \rangle \\
&\stackrel{(3.76)}{=} \sum_{|\alpha| \leq \omega} \langle {}^0t, x^\alpha (\mathbb{I} - g) (V_{\omega-|\alpha|} w) \rangle \frac{\partial_\alpha f(0)}{\alpha!} \\
&\doteq \sum_{|\alpha| \leq \omega} b^\alpha(g) \frac{\partial_\alpha f(0)}{\alpha!}.
\end{aligned} \tag{3.77}$$

Then (3.77) defines a map from  $G$  to a finite dimensional complex vector space. Now we follow [SP82], [Sch95][chapter 4.5]: Applying two transformations

$$b^\alpha(g_1 g_2) = \langle {}^0t, x^\alpha (\mathbb{I} - g_1 g_2) (V_{\omega-|\alpha|} w) \rangle \tag{3.78}$$

$$= \langle {}^0t, x^\alpha ((\mathbb{I} - g_1) + g_1 (\mathbb{I} - g_2)) (V_{\omega-|\alpha|} w) \rangle \tag{3.79}$$

$$= b^\alpha(g_1) + |\det g_1| \langle {}^0t(g_1 x), (g_1 x)^\alpha (\mathbb{I} - g_2) (V_{\omega-|\alpha|} w) \rangle, \tag{3.80}$$

and omitting the indices we see  $b(g_1 g_2) = b(g_1) + D(g_1) g_1 b(g_2)$ , which is a 1-cocycle for  $b(g)$ . Its trivial solutions are the 1-coboundaries

$$b(g) = (\mathbb{I} - D(g)g)a, \tag{3.81}$$

and these are the only ones if the first cohomology group of  $G$  is zero. In that case we can restore  $G$ -covariance by adding the following counter terms:

$$\langle t_{(\omega;w)}^{G-\text{cov}}, f \rangle \doteq \langle t_{(\omega;w)}, f \rangle + \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} a^\alpha(w) \partial_\alpha f(0). \tag{3.82}$$

The task is to determine  $a$  from (3.81) and (3.77):

$$\langle {}^0t, x^\alpha (\mathbb{I} - g) (V_{\omega-|\alpha|} w) \rangle = [(\mathbb{I} - D(g)g)a]^\alpha \tag{3.83}$$

**5.3. Bosonic Lorentz covariance.** The first cohomology group of  $\mathcal{L}_+^\dagger$  vanishes [Sch95][chapter 4.5 and references there]. We determine  $a$  from the last equation. The most simple solution appears in the case of Lorentz invariance in one coordinate. This situation was completely analyzed in [BPP99] for  $\partial_\alpha w(0) = \delta_\alpha^0$ . The following two subsections generalize the results to arbitrary  $w$ ,  $w(0) \neq 0$ .

5.3.1. *Lorentz invariance in  $\mathbb{M}$ .* If we expand the index  $\alpha$  into Lorentz indices  $\mu_1, \dots, \mu_n$ , (3.83) is symmetric in  $\mu_1, \dots, \mu_n$  and therefore  $a$  is, too. We just state our result from [BPP99] which is modified by the generalization of  $w$ :

$$a^{(\mu_1 \dots \mu_n)} = \frac{(n-1)!!}{(n+2)!!} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-2s)!!}{(n-2s-1)!!} \eta^{(\mu_1 \mu_2} \dots \eta^{\mu_{2s-1} \mu_{2s}} \times \\ \times \left\langle {}^0 t, (x^2)^s x^{\mu_{2s+1}} \dots x^{\mu_{n-1}} \left( x^2 \partial^{\mu_n} - x^{\mu_n} x^\beta \partial_\beta \right) V_{\omega-n} w \right\rangle, \quad (3.84)$$

if we choose the fully contracted part of  $a$  to be zero in case of  $n$  being even. We used the notation

$$b^{(\mu_1 \dots \mu_n)} = \frac{1}{n!} \sum_{\pi \in S_n} b^{\mu_{\pi(1)} \dots \mu_{\pi(n)}}, \quad b^{[\mu_1 \dots \mu_n]} = \frac{1}{n!} \sum_{\pi \in S_n} \text{sgn}(\pi) b^{\mu_{\pi(1)} \dots \mu_{\pi(n)}},$$

for the totally symmetric resp. antisymmetric part of a tensor.

5.3.2. *Dependence on  $w$ .* Performing a functional derivation of the Lorentz invariant extension with respect to  $w$ , only Lorentz invariant counter terms appear.

DEFINITION 2. The functional derivation is given by:

$$\left\langle \frac{\delta}{\delta h} F(h), f \right\rangle \doteq \frac{d}{d\lambda} F(h + \lambda f) \Big|_{\lambda=0}. \quad (3.85)$$

This definition implies the following functional derivatives:

$$\left\langle \frac{\delta}{\delta w} t_{(\omega;w)}(f), h \right\rangle = - \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle t_{(\omega;w)}, x^\alpha h \rangle \partial_\alpha (f w^{-1})(0), \quad (3.86)$$

$$\left\langle \frac{\delta}{\delta w} \langle s, V_\omega w \rangle, h \right\rangle = \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle s, W_{(\omega;w)}(x^\alpha h) \rangle \partial_\alpha w^{-1}(0), \quad (3.87)$$

for any distribution  $s \in \mathcal{D}'(\mathbb{R}^d)$  and  $f, h \in \mathcal{D}(\mathbb{R}^d)$ .

PROOF. We show how to derive the first relation. Inserting the definition we find:

$$\begin{aligned}
& \left. \frac{d}{d\lambda} t_{(\omega; w + \lambda h)}(f) \right|_{\lambda=0} = \\
& = \left\langle {}^0t, -h \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \partial_\alpha (f w^{-1})(0) + w \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \partial_\alpha (f h w^{-2})(0) \right\rangle, \\
& = \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \left\langle {}^0t, -h x^\alpha \partial_\alpha (f w^{-1})(0) + \right. \\
& \quad \left. + w x^\alpha \partial_\alpha (f w^{-1})(0) \sum_{|\nu| \leq \omega - |\alpha|} \frac{x^\nu}{\nu!} \partial_\nu (h w^{-1})(0) \right\rangle \quad (3.88) \\
& = - \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle {}^0t, x^\alpha W_{(\omega - |\alpha|; w)} h \rangle \partial_\alpha (f w^{-1})(0) \\
& = - \sum_{|\alpha| \leq \omega} \frac{1}{\alpha!} \langle t_{(\omega; w)}, x^\alpha h \rangle \partial_\alpha (f w^{-1})(0),
\end{aligned}$$

where we used Leibnitz rule and rearranging of the summation of the second term in the second line and the relation  $x^\alpha W_{(\omega - |\alpha|; w)} f = W_{(\omega; w)}(x^\alpha f)$  on the last line. The second equation follows from a similar calculation.  $\square$

We calculate the dependence of  $a$  on  $w$ . With (3.87) we get:

$$\begin{aligned}
& \left\langle \frac{\delta}{\delta w} a^{(\mu_1 \dots \mu_n)}(w), f \right\rangle = \frac{(n-1)!!}{(n+2)!!} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-2s)!!}{(n-1-2s)!!} \eta^{(\mu_1 \mu_2 \dots \mu_{2s-1} \mu_{2s})} \times \\
& \times \sum_{|\beta| \leq \omega - n} \frac{\partial_\beta w^{-1}(0)}{\beta!} \left\langle {}^0t, (x^2)^s x^{\mu_{2s+1}} \dots x^{\mu_{n-1}} (x^2 \partial^{\mu_n} - x^{\mu_n}) x^\beta \partial_\beta \right\rangle W_{(\omega - n; w)}(x^\beta f) \rangle. \quad (3.89)
\end{aligned}$$

Since  $W_{(\omega - n; w)}(x^\beta h)$  is sufficient regular, we can put the  $x$ 's and derivatives on the left and the same calculation like in [BPP99] applies. The result is

$$\begin{aligned}
& \left\langle \frac{\delta}{\delta w} a^{\mu_1 \dots \mu_n}(w), f \right\rangle = \sum_{|\beta| \leq \omega - n} \frac{\partial_\beta w^{-1}(0)}{\beta!} \left[ \left\langle t_{(\omega; w)}, x^{\mu_1} \dots x^{\mu_n} x^\beta f \right\rangle + \right. \\
& \quad \left. - \begin{cases} 0, & n \text{ odd,} \\ \frac{2(n-1)!!}{(n+2)!!} \left\langle t_{(\omega; w)}, (x^2)^{\frac{n}{2}} x^\beta f \right\rangle \eta^{(\mu_1 \mu_2 \dots \mu_{n-1} \mu_n)}, & n \text{ even.} \end{cases} \right] \quad (3.90)
\end{aligned}$$



Using this result and (3.86) we find:

$$\left\langle \frac{\delta}{\delta w} \left\langle t_{(\omega;w)}^{\text{linv}}, f \right\rangle, h \right\rangle = - \sum_{\substack{n=0 \\ n \text{ even}}}^{\omega} \frac{d_n}{n!} \square^{\frac{n}{2}} f(0), \quad (3.91)$$

$$d_n \doteq \frac{2(n-1)!!}{(n+2)!!} \sum_{|\beta| \leq \omega-n} \frac{1}{\beta!} \left\langle t_{(\omega;w)}, (x^2)^{\frac{n}{2}} x^\beta h \right\rangle \partial_\beta w^{-1}(0), \quad (3.92)$$

where we set  $d_0 = 1$ .

**5.4. General Lorentz covariance.** If the distribution  ${}^0t$  depends on more than one variable,  ${}^0tx^\alpha$  is not symmetric in all Lorentz indices in general. Since  $x^\alpha$  transforms like a tensor, it is natural to generalize the discussion to the case, where  ${}^0t$  transforms like a tensor, too. Assume  $\text{rank}({}^0t) = r$ , then  $D(g)g$  is the tensor representation of rank  $p = r + n$ ,  $n = |\alpha|$ , in (3.83). From now on we omit the indices. So if  $t \in \mathcal{D}(\mathbb{M}^m \setminus \{0\})$ , we denote by  $\tilde{x}$  – formerly  $x^\alpha$  – a tensor of rank  $n$  built of  $x_1, \dots, x_m$ .

To solve (3.83) we proceed like in [BPP99]. Since the equation holds for all  $g$  we solve for  $a$  by using Lorentz transformations in the infinitesimal neighbourhood of  $\mathbb{I}$ . If we take  $\theta_{\alpha\beta} = \theta_{[\alpha\beta]}$  as six coordinates these transformations read:

$$g \approx \mathbb{I} + \frac{1}{2} \theta_{\alpha\beta} l^{\alpha\beta}, \quad (3.93)$$

with the generators

$$(l^{\alpha\beta})^\mu{}_\nu = \eta^{\alpha\mu} \delta_\nu^\beta - \eta^{\beta\mu} \delta_\nu^\alpha. \quad (3.94)$$

Then, for an infinitesimal transformation one finds from (3.83):

$$B^{\alpha\beta} \doteq 2 \left\langle {}^0t, \tilde{x} \sum_{j=1}^m x_j^{[\alpha} \partial_j^{\beta]} (V_{\omega-n} w) \right\rangle = (l^{\alpha\beta} \otimes \dots \otimes \mathbb{I} + \dots + \mathbb{I} \otimes \dots \otimes l^{\alpha\beta}) a, \quad (3.95)$$

$\alpha, \beta$  being Lorentz four-indices. In [BPP99] our ability to solve that equation heavily relied on the given symmetry, which is in general absent here. Nevertheless we can find an inductive construction for  $a$ , corresponding to equation (29) in [BPP99].

We build one Casimir operator on the r.h.s. (the other one is always zero, since we are in a  $(1/2, 1/2)^{\otimes p}$  representation).

*The case  $p = 1$ .* Just to remind that  $p$  is the rank of  $\tilde{x}t$ , this occurs if either  $t$  is a vector and  $\tilde{x} = 1, (n = 0)$ , or  $t$  is a scalar and  $\tilde{x} = x_1, \dots, x_m$ . (3.95) gives:

$$\frac{1}{2} l_{\alpha\beta} B^{\alpha\beta} = \frac{1}{2} l_{\alpha\beta} l^{\alpha\beta} a = -3\mathbb{I}a, \quad (3.96)$$

since the Casimir operator is diagonal in the irreducible  $(1/2, 1/2)$  representation.

The case  $p = 2$ . We get

$$\frac{1}{2}(l_{\alpha\beta} \otimes \mathbb{I} + \mathbb{I} \otimes l_{\alpha\beta})B^{\alpha\beta} = (-6\mathbb{I} + l_{\alpha\beta} \otimes l^{\alpha\beta})a. \quad (3.97)$$

Since  $a$  is a tensor of rank 2, let us introduce the projector onto the symmetric resp. antisymmetric part and the trace:

$$P_S^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}(\delta_\rho^\mu \delta_\sigma^\nu + \delta_\rho^\nu \delta_\sigma^\mu), \quad P_A^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu), \quad P_\eta^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}\eta^{\mu\nu}\eta_{\rho\sigma}, \quad (3.98)$$

$$P^2 = P, \quad P_S + P_A = \mathbb{I}, \quad P_S - P_A = \tau, \quad (3.99)$$

where  $\tau$  denotes the permutation of the two indices. Using (3.94), we find

$$\frac{1}{2}l_{\alpha\beta} \otimes l^{\alpha\beta} = 4P_\eta - \tau. \quad (3.100)$$

Now we insert (3.100) into (3.97). The trace part is set to zero again. Acting with  $P_A$  and  $P_S$  on the resulting equation gives us two equations for the antisymmetric and symmetric part respectively. This yields:

$$a = -\frac{1}{16}(P_S + 2P_A)(l_{\alpha\beta} \otimes \mathbb{I} + \mathbb{I} \otimes l_{\alpha\beta})B^{\alpha\beta}. \quad (3.101)$$

*Inductive assumption.* Now we turn back to equation (3.83). We note that any contraction commutes with the (group) action on the rhs. Hence, if we contract (3.95), we find on the rhs:

$$\eta_{ij}(l^{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes l^{\alpha\beta})a = (l^{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \not{j} + \cdots + \not{i} + \cdots + \mathbb{I} \otimes \cdots \otimes l^{\alpha\beta})(\eta_{ij}a),$$

where  $i, j$  denote the positions of the corresponding indices. Therefore the rank of (3.95) is reduced by two and we can proceed inductively. With the cases  $p = 1, p = 2$  solved, we assume that all possible contractions of  $a$  are known.

*Induction step.* Multiplying (3.95) with the generator and contracting the indices yields:

$$\left(3p\mathbb{I} + 2 \sum_{\tau \in S_p} \tau\right)a = -\frac{1}{2}(l_{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes l_{\alpha\beta})B^{\alpha\beta} + 8 \sum_{i < j \leq p} P_{\eta_{ij}}a. \quad (3.102)$$

The transposition  $\tau$  acts on  $a$  by permutation of the corresponding indices. For a general  $\pi \in S_p$  the action on  $a$  is given by:  $\pi a^{\mu_1 \cdots \mu_p} = a^{\mu_{\pi^{-1}(1)} \cdots \mu_{\pi^{-1}(p)}}$ . In order to solve this equation we consider the representation of the symmetric groups. We give a brief summary of all necessary ingredients in appendix A. So let  $k_\tau \doteq \sum_{\tau \in S_p} \tau$  be the sum of all transpositions of  $S_p$ . Then  $k_\tau$  is in the center of the group algebra  $\mathcal{A}_{S_p}$ . It can be decomposed into the idempotents  $e_{(m)}$  that generate the irreducible representations of  $S_p$  in  $\mathcal{A}_{S_p}$ .

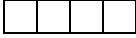
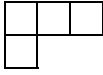

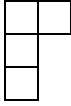
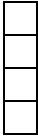
$$k_\tau = h_\tau \sum_{(m)} \frac{1}{f_{(m)}} \chi_{(m)}(\tau) e_{(m)}. \quad (3.103)$$

The sum runs over all partitions  $(m) = (m_1, \dots, m_r)$ ,  $\sum_{i=1}^r m_i = p$ ,  $m_1 \geq m_2 \geq \dots \geq m_r$  and  $h_\tau = \frac{1}{2}p(p-1)$  is the number of transpositions in  $S_p$ .  $\chi_{(m)}$  is the character of  $\tau$  in the representation generated by  $e_{(m)}$  which is of dimension  $f_{(m)}$ . We use (3.103), the orthogonality relation  $e_{(m)}e_{(m')} = \delta_{(m)(m')}$  and the completeness  $\sum_{(m)} e_{(m)} = \mathbb{I}$  in (3.102). The expression in brackets on the l.h.s may be orthogonal to some  $e_{(m)}$ . The corresponding  $e_{(m)}a$  contribution is any combinations of  $\eta$ 's and  $\epsilon$ 's  $-\epsilon$  being the totally antisymmetric tensor in four dimensions – transforming invariantly and thus can be set to zero. We arrive at

$$a = \sum_{\substack{(m) \\ c(m) \neq 0}} \frac{e_{(m)}}{c(m)} \left( -\frac{1}{2} (l_{\alpha\beta} \otimes \dots \otimes \mathbb{I} + \dots + \mathbb{I} \otimes \dots \otimes l_{\alpha\beta}) B^{\alpha\beta} + 8 \sum_{i < j \leq p} P_{\eta_{ij}} a \right),$$

$$c(m) \doteq 3p + p(p-1) \frac{\chi_{(m)}(\tau)}{f_{(m)}} = 3p + \sum_{i=1}^r \left( b_i^{(m)} (b_i^{(m)} + 1) - a_i^{(m)} (a_i^{(m)} + 1) \right), \quad (3.104)$$

with  $a = (a_1, \dots, a_r)$ ,  $b = (b_1, \dots, b_r)$  denoting the characteristics of the frame  $(m)$ , see appendix A. Let us take  $p = 4$  as an example:

idempotent	Young frame	dimension	character
$e_{(4)}$		$f_{(4)} = 1$	$\chi_{(4)}(\tau) = 1$
$e_{(3,1)}$		$f_{(3,1)} = 3$	$\chi_{(3,1)}(\tau) = 1$
$e_{(2,2)}$		$f_{(2,2)} = 2$	$\chi_{(2,2)}(\tau) = 0$
$e_{(2,1,1)}$		$f_{(2,1,1)} = 3$	$\chi_{(2,1,1)}(\tau) = -1$
$e_{(1,1,1,1)}$		$f_{(1,1,1,1)} = 1$	$\chi_{(1,1,1,1)}(\tau) = -1$

We find for (3.102)

$$a = \frac{1}{48} (2e_{(4)} + 3e_{(3,1)} + 4e_{(2,2)} + 6e_{(2,1,1)}) \times \text{r.h.s}(3.102). \quad (3.105)$$

We see that no  $e_{(1,1,1,1)}$  appears in that equation. It corresponds to the one dimensional *sgn*-representation of  $S_4$ , so  $e_4 a \propto \epsilon$ .

**5.5. Spinorial Lorentz covariance.** This subsection uses the conventions of [SU92]. Any finite dimensional representation of  $\mathcal{L}_+^\uparrow$  can be reduced to tensor

products of  $SL(2, \mathbb{C})$  and  $\overline{SL(2, \mathbb{C})}$  and direct sums of these. A two component spinor  $\Psi$  transforms according to

$$\Psi^A = g^A_B \Psi^B, \quad (3.106)$$

where  $g$  is a  $2 \times 2$ -matrix in the  $SL(2, \mathbb{C})$  representation of  $\mathcal{L}_+^\dagger$ . For the complex conjugated representation we use the dotted indices, i.e.

$$\overline{\Psi}^{\dot{X}} = \overline{g}^{\dot{X}}_{\dot{Y}} \overline{\Psi}^{\dot{Y}}, \quad (3.107)$$

with  $\overline{g}^{\dot{X}}_{\dot{Y}} = \overline{g^X_Y}$  in the  $\overline{SL(2, \mathbb{C})}$  representation. The indices are lowered and raised with the  $\epsilon$ -tensor.

$$\epsilon_{AB} = \overline{\epsilon}_{\dot{A}\dot{B}} \doteq \epsilon_{\dot{A}\dot{B}}, \quad (3.108)$$

$$\epsilon^{AB} \epsilon_{AC} = \epsilon^{BA} \epsilon_{CA} = \delta_C^B. \quad (3.109)$$

We define the Van-der-Waerden symbols with the help of the Pauli matrices  $\sigma_\mu$  and  $\tilde{\sigma}_\mu \doteq \sigma^\mu$ :

$$\sigma_\mu^{A\dot{X}} \doteq \frac{1}{\sqrt{2}}(\sigma_\mu)^{AX}, \quad \sigma_{\mu A\dot{X}} \doteq \frac{1}{\sqrt{2}}(\tilde{\sigma}_\mu^T)_{AX}. \quad (3.110)$$

They satisfy the following relations

$$\sigma_\mu^{A\dot{X}} \sigma_{\nu A\dot{X}} = \eta_{\mu\nu} \quad \sigma_{\mu A\dot{X}} \sigma^\mu_{B\dot{Y}} = \epsilon_{AB} \epsilon_{\dot{X}\dot{Y}} \quad (3.111)$$

With the help of these we can build the infinitesimal spinor transformations

$$g \approx \mathbb{I} + \frac{1}{2} \theta_{\alpha\beta} S^{\alpha\beta}, \quad (3.112)$$

with the generators

$$(S^{\alpha\beta})^A_B = \sigma^{[\alpha A\dot{X}} \sigma^{\beta]}_{B\dot{X}}. \quad (3.113)$$

Note that the  $\sigma$ 's are hermitian:  $\overline{\sigma_\mu^{A\dot{X}}} = \sigma_\mu^{X\dot{A}}$ . Again we define the projectors for the tensor product. But we have only two irreducible parts:

$$P_S^{AB}{}_{CD} = \frac{1}{2}(\delta_C^A \delta_D^B + \delta_D^A \delta_C^B), \quad P_\epsilon^{AB}{}_{CD} = \frac{1}{2}\epsilon^{AB} \epsilon_{CD}, \quad (3.114)$$

$$P_S^2 = P_S, \quad P_\epsilon^2 = P_\epsilon, \quad P_S + P_\epsilon = \mathbb{I}. \quad (3.115)$$

We get the following identities:

$$S^{\alpha\beta} S_{\alpha\beta} = \overline{S}^{\alpha\beta} \overline{S}_{\alpha\beta} = -3\mathbb{I}, \quad (3.116)$$

$$S^{\alpha\beta} \otimes S_{\alpha\beta} = \overline{S}^{\alpha\beta} \otimes \overline{S}_{\alpha\beta} = 4P_\epsilon - \mathbb{I}, \quad (3.117)$$

$$S^{\alpha\beta} \otimes \overline{S}_{\alpha\beta} = \overline{S}^{\alpha\beta} \otimes S_{\alpha\beta} = 0. \quad (3.118)$$

In order to have (3.83) in a pure spinor representation we have to decompose the tensor  $\tilde{x}$  into spinor indices according to

$$x^{A\dot{X}} \doteq x^\mu \sigma_\mu^{A\dot{X}}. \quad (3.119)$$

Assume  $t\tilde{x}$  transforms under the  $u$ -fold tensor product of  $SL(2, \mathbb{C})$  times the  $v$ -fold tensor product of  $\overline{SL(2, \mathbb{C})}$  then, for infinitesimal transformations, (3.83) yields:

$$B^{\alpha\beta} = (S^{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \overline{S}^{\alpha\beta})a, \quad (3.120)$$

with  $B^{\alpha\beta}$  from equation (3.95) in the corresponding spinor representation. The sum consists of  $u$  summands with one  $S^{\alpha\beta}$  and  $v$  summands with one  $\overline{S}^{\alpha\beta}$  with  $u, v > n$ . Multiplying again with the generator and contracting the indices gives twice the Casimir on the r.h.s. Inserting (3.116-3.118) yields:

$$\begin{aligned} & (S_{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \overline{S}_{\alpha\beta})B^{\alpha\beta} \\ &= \left( -3(u+v)\mathbb{I} + 2 \sum_{1 \leq i < j \leq u} (4P_{\epsilon_{ij}} - \mathbb{I}) + 2 \sum_{1 \leq i < j \leq v} (4P_{\bar{\epsilon}_{ij}} - \mathbb{I}) \right) a. \end{aligned} \quad (3.121)$$

The sum over  $u$  runs over  $\frac{u}{2}(u-1)$  possibilities and similar for  $v$ , so we find the induction:

$$\begin{aligned} a = \frac{1}{u(u+2) + v(v+2)} & \left[ -(S_{\alpha\beta} \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \overline{S}_{\alpha\beta})B^{\alpha\beta} + \right. \\ & \left. 8 \left( \sum_{1 \leq i < j \leq u} P_{\epsilon_{ij}} + \sum_{1 \leq i < j \leq v} P_{\bar{\epsilon}_{ij}} \right) a \right]. \end{aligned} \quad (3.122)$$

It already contains the induction start for  $a^{(AB)}$ ,  $a^{(XY)}$  and  $a^{A\dot{X}}$ .

**5.6. General covariant BPHZ subtraction.** We have derived a Lorentz covariant renormalization that applies for a general choice of  $w$ . Therefore, choosing  $e^{ip\cdot}$  as test function provides for a covariant renormalization in momentum space. The choice  $w = e^{iq\cdot}$  corresponds to subtraction at momentum  $q$  [Pra99a]. Hence  $q = 0 \Leftrightarrow w = 1$  represents BPHZ subtraction.

But this choice leads to infrared divergencies in massless theories. Lowenstein and Zimmermann [LZ75] have introduced an alternative scheme (called BPHZL) that makes use of an auxiliary mass and requires additional subtractions with respect to a parameter which scales this mass. This also produces a covariant renormalization in momentum space. We compare their results with ours in two examples.

We shrink the distribution space to  $\mathcal{S}'$  since we are dealing with Fourier transformation. Let  $x, q, p \in \mathbb{M}^m$ ,

$$\widehat{t_{(\omega; e^{iq\cdot})}}(p) \doteq \langle t_{(\omega; e^{iq\cdot})}, e^{ip\cdot} \rangle \quad (3.123)$$

$$= \left\langle {}^0t, e^{ip\cdot} - \sum_{|\alpha| \leq \omega} \frac{(p-q)^\alpha}{\alpha!} \partial_\alpha^q e^{iq\cdot} \right\rangle. \quad (3.124)$$

It is normalized at the subtraction point  $q$ , i.e.:  $\partial_\alpha \widehat{t_{(\omega; e^{iq\cdot})}}(q) = 0$ ,  $|\alpha| \leq \omega$ . This is always possible for  $q$  totally space like,  $(\sum_{j \in I} q_j)^2 < 0, \forall I \subset \{1, \dots, m\}$  [EG73, Düt]. If we use the results from above we obtain a covariant BPHZ subtraction at momentum  $q$  by adding  $\sum_{|\alpha| \leq \omega} \frac{i^{|\alpha|}}{\alpha!} a^\alpha p_\alpha$  to (3.124), according to

equation (3.82). For  $|\beta| \geq \omega + 1$ ,  ${}^0tx^\beta$  is a well defined distribution on  $\mathbb{S}$  and so is  $\partial_\beta \widehat{{}^0t}$ .

5.6.1. *Lorentz invariance on  $\mathbb{M}$ .* We have

$$V_k e^{iq \cdot} = e^{iq \cdot} \sum_{m=0}^k \frac{1}{m!} (-ipx)^m, \quad \partial_\sigma V_k e^{iq \cdot} = iq_\sigma e^{iq \cdot} \frac{1}{k!} (-ipx)^k. \quad (3.125)$$

Inserting this into (3.84) we find:

$$\begin{aligned} a^{(\mu_1 \dots \mu_n)} &= \\ &= \frac{i^n (-)^{\omega+1}}{(\omega-n)!} \frac{(n-1)!!}{(n+2)!!} q_{\sigma_1} \dots q_{\sigma_{\omega-n}} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-2s)!!}{(n-2s-1)!!} \left( q_\rho \partial^\rho \partial^{(\mu_1} - q^{(\mu_1} \square \right) \times \\ &\quad \times \eta^{\mu_2 \mu_3} \dots \eta^{\mu_{2s} \mu_{2s+1}} \partial^{\mu_{2s+2}} \dots \partial^{\mu_n)} \square^s \partial^{\sigma_1} \dots \partial^{\sigma_{\omega-n}} \widehat{{}^0t}(q). \end{aligned} \quad (3.126)$$

Let us consider two examples namely the fish and the setting sun from massless scalar field theory. We compare the results to BPHZL [LZ75].

EXAMPLE 1. The fish graph corresponds to  ${}^0t = \frac{i^2}{2} D_F^2 \Rightarrow \omega = 0$ . It needs no counter term, since it is Lorentz invariant. We translate the result of BPHZL to coordinate space:

$$\widehat{(D^F)^2}_{\text{BPHZL}}(p) = \left\langle (D^F)^2 - (\Delta^F)^2, e^{ip \cdot} \right\rangle + \left\langle (\Delta^F)^2, W_{(0;1)} e^{ip \cdot} \right\rangle. \quad (3.127)$$

EXAMPLE 2. Take the setting sun in massless scalar field theory:  ${}^0t = \frac{i^3}{6} D_F^3 \Rightarrow \omega = 2$ .

$$\begin{aligned} a^\mu &= -\frac{i}{3} (q_\sigma q_\rho \partial^\rho \partial^\sigma \partial^\mu - q^\mu q_\sigma \partial^\sigma \square) \widehat{{}^0t}(q), \\ a^{\mu\nu} &= \frac{1}{4} (q_\rho \partial^\rho \partial^\mu \partial^\nu - q^{(\mu} \partial^{\nu)} \square) \widehat{{}^0t}(q), \end{aligned}$$

and adding  $ip_\mu a^\mu - \frac{1}{2} p_\mu p_\nu a^{\mu\nu}$  restores Lorentz invariance of the setting sun graph subtracted at  $q$ . Renormalization according to BPHZL gives:

$$\begin{aligned} \widehat{(D^F)^3}_{\text{BPHZL}}(p) &= \left\langle (D^F)^3 - (\Delta^F)^3, W_{(1;1)} e^{ip \cdot} \right\rangle + \\ &\quad + \left\langle (\Delta^F)^3, W_{(2;1)} e^{ip \cdot} \right\rangle. \end{aligned} \quad (3.128)$$

**5.7. General induction.** We only have to evaluate  $B^{\alpha\beta}$  with  $w = e^{iq \cdot}$  and plug the result into the induction formulas (3.122) and (3.104).

$$B^{\alpha\beta} = 2i^n (-)^{\omega+1} \sum_{j=1}^m \sum_{|\gamma|=\omega-n} \frac{q_j^\gamma}{\gamma!} q_j^{[\alpha} \partial_j^{\beta]} \partial_\gamma \tilde{\partial} \widehat{{}^0t}(q). \quad (3.129)$$

Here,  $q_j$  are the  $m$  components of  $q$  hence  $\gamma$  is a  $4m$  index and  $\alpha, \beta$  are four indices. The tensor (spinor) structure of  $\tilde{\partial}$  is given by  $\tilde{x}$  in (3.95).

## CHAPTER 4

### Local perturbative interacting fields

In the previous chapters we have introduced free quantum fields, Wick products of these fields and finally time ordered products of Wick polynomials. It turns out that this provides a complete frame for the definition of interacting (perturbative) fields.<sup>1</sup>

The  $S$ -matrix serves as the generating functional for time ordered products of the interaction by smearing with a test function of compact support. After coupling additional fields into the  $S$ -matrix one obtains interacting fields and time ordered products of them by Bogoliubov's formula [BS76].

This situation provides a setting for the introduction of local observable algebras: We choose a causally complete bounded spacetime region  $\mathcal{O}$  on which the coupling is assumed to be constant. It was shown by Brunetti and Fredenhagen [BF96] that the interacting fields on this region only change by a unitary transformation if the interaction is changed outside  $\mathcal{O}$ . Therefore, algebraic relations are left invariant.

Let us emphasize that the occurrence of IR-singularities is a consequence of performing the *adiabatic limit* where the test function of the interaction tends to a constant. Since we avoid this limit it is always possible to construct the local algebras. Especially for asymptotic free theories like Yang-Mills for example [Boa99] this is an advantage. Here perturbation theory can be regarded as a valid approximation only for short distances. The involved fields on these scales do not correspond to asymptotic particles that can be observed in experiments.

Although our work only deals with perturbative fields we remark that Bogoliubov's formula for interacting fields also applies for the non perturbative case.

#### 1. The $S$ -matrix

For a symbol  $\mathcal{L} \in \mathfrak{B}$  describing the interaction :  $\mathcal{L} : \in \text{Dist}_1(\mathcal{D})$  of our quantum theory we build the  $S$ -matrix:

$$S(g\mathcal{L}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n T(g\mathcal{L}, \dots, g\mathcal{L})(x_1, \dots, x_n), \quad (4.1)$$

where we allowed for the use of  $g\mathcal{L}$  as an argument in the  $T$ -product as explained in **P1**. We also consider the case of a sum of different couplings, such that  $g\mathcal{L} = g \cdot \mathcal{L} = \sum_i^s g_i \mathcal{L}_i$ . The  $S$ -matrix is the generating functional for all

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<sup>1</sup>Here and in the following the name “field” also includes composed objects, which are Wick polynomials in the free case.

$T$ -products of the  $\mathcal{L}_i$ :

$$T(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) = \frac{\delta^n}{i^n \delta g_{i_1}(x_1) \dots \delta g_{i_n}(x_n)} S(g\mathcal{L}) \Big|_{g_1=\dots=g_s=0}. \quad (4.2)$$

The inverse of  $S$  is given by

$$S(g\mathcal{L})^{-1} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \dots dx_n \bar{T}(g\mathcal{L}, \dots, g\mathcal{L})(x_1, \dots, x_n) \quad (4.3)$$

which follows from (3.50) by the inversion of a formal power series. Then  $S$  is a unitary operator on  $\mathcal{D}$ :

$$S(g\mathcal{L})^{-1} = S(g\mathcal{L})^*, \quad (4.4)$$

because of normalization condition **N2**. Obviously  $S^{-1}$  is the generating functional of the  $\bar{T}$ -products:

$$\bar{T}(\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_n})(x_1, \dots, x_n) = \frac{\delta^n}{(-i)^n \delta g_{i_1}(x_1) \dots \delta g_{i_n}(x_n)} S(g\mathcal{L})^{-1} \Big|_{g_1=\dots=g_s=0}. \quad (4.5)$$

Because of the causal factorization of the  $T$ -products the  $S$ -matrix fulfils the following causal factorization:

$$S(fW + gL + hV) = S(fW + gL)S(gL)^{-1}S(gL + hV), \quad (4.6)$$

$\forall f, g, h \in \mathcal{D}(\mathbb{M}), W, L, V \in \mathfrak{B}$ , if  $\text{supp } f \cap \bar{V}_-(\text{supp } h) = \emptyset$  as was shown in [EG73]. Especially in the case  $g = 0$  this leads to

$$S(fW + hV) = S(fW)S(hV), \quad (4.7)$$

which is the causal factorization of the  $T$ -products lifted to the functionals.

## 2. Interacting fields

We now couple additional source terms into the local  $S$ -matrix defined above. This generates a new functional called the *relative*  $S$ -matrix according to:

$$S_{g\mathcal{L}}(hW) \doteq S(g\mathcal{L})^{-1}S(g\mathcal{L} + hW). \quad (4.8)$$

The relative  $S$ -matrix also satisfies the causal factorization equations (4.6), (4.7). Especially the latter one reads

$$S_{g\mathcal{L}}(h_1W_1 + h_2W_2) = S_{g\mathcal{L}}(h_1W_1)S_{g\mathcal{L}}(h_2W_2) = S_{g\mathcal{L}}(h_2W_2)S_{g\mathcal{L}}(h_1W_1), \quad (4.9)$$

if  $\text{supp } h_1 \sim \text{supp } h_2$  and  $W_1, W_2 \in \mathfrak{B}$ . Hence the relative  $S$ -matrices commute for space like coupled sources [BF00] and may therefore serve as generating functionals for local fields. Then the *interacting field*  $W_{g\mathcal{L}}$  corresponding to the symbol  $W \in \mathfrak{B}$  is defined by Bogoliubov's formula [BS76, EG73]:

$$W_{g\mathcal{L}}(x) \doteq \frac{\delta}{i\delta h(x)} S_{g\mathcal{L}}(hW) \Big|_{h=0}. \quad (4.10)$$



Expanding the interacting field (4.10) into a power series results in

$$W_{g\mathcal{L}}(x) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_1 \dots dy_n R(g\mathcal{L}, \dots, g\mathcal{L}; W)(y_1, \dots, y_n; x), \quad (4.11)$$

where the *retarded products* ( $R$ -products) are given by

$$R(W_1, \dots, W_{(n)}; W)(y_1, \dots, y_n; x) \doteq \sum_{I \subset N} (-)^{|I|} \overline{T}(I)(y_I) T(I^c, W)(y_{I^c}, x). \quad (4.12)$$

With the help of equation (3.50),  $x$  can also be put in the  $\overline{T}$ -products, yielding

$$= \sum_{I \subset N} (-)^{|I|} \overline{T}(I, W)(y_I, x) T(I^c)(y_{I^c}). \quad (4.13)$$

The word retarded encodes the support properties of the  $R$ -products:

$$\begin{aligned} \text{supp } R(W_1, \dots, W_n; W)(y_1, \dots, y_n; x) \subset \\ \subset \{(y_1, \dots, y_n, x) \in \mathbb{M}^{n+1}, y_i \in \overline{V}_-(x), \forall i = 1, \dots, n\}. \end{aligned} \quad (4.14)$$

This can be seen immediately from the causality property **P3** [EG73].

**2.1. Properties of the interacting fields.** The properties of the time ordered products **P1** – **P4** and the normalization conditions **N1** – **N4** have immediate consequences for the interacting fields defined above. Because of (3.50) and the definition of  $R$  we find that

$$\mathbb{I}_{g\mathcal{L}} = \mathbb{I}. \quad (4.15)$$

Since the interaction  $\mathcal{L}$  is assumed to be a Lorentz scalar, the normalization condition **N1** implies the conservation of the Poincaré transformation properties:

$$(\text{Ad}U(L))W_{g\mathcal{L}}(x) = (D(\Lambda^{-1})(W))_{Lg\mathcal{L}}(Lx), \forall L = (a, \Lambda) \in \mathcal{P}_+^\uparrow, \quad (4.16)$$

where  $D$  is the representation of  $\mathcal{L}_+^\uparrow$  according to (3.7),(3.8).  $\mathcal{P}_+^\uparrow$  acts on  $\mathcal{D}(\mathbb{M})$  as a group homomorphism according to  $Lf(x) = f(L^{-1}x)$ ,  $f \in \mathcal{D}(\mathbb{M})$ . The  $*$ -involution on the interacting fields is given by

$$(W_{g\mathcal{L}})^* = (W^*)_{g\mathcal{L}}, \quad (4.17)$$

where on the LHS  $*$  is the adjoint on  $\mathcal{D}$  and on the RHS it is given by (3.9),(3.10). Since the generating functionals commute for spacelike separated sources according to (4.9) the interacting fields are local:

$$[W_{g\mathcal{L}}(x), V_{g\mathcal{L}}(y)] = 0, \text{ if } x \sim y. \quad (4.18)$$

Normalization condition **N4** further implies the interacting equations of motion for the interacting basic generator.

$$D_{ij}\varphi_{jg\mathcal{L}} = - \left( \frac{\partial \mathcal{L}}{\partial \varphi_i} \right)_{g\mathcal{L}}, \quad \varphi_j \in \mathfrak{G}_b. \quad (4.19)$$

The commutator of two interacting fields was calculated in [DF99]:

$$\begin{aligned} [W_{g\mathcal{L}}(x), V_{g\mathcal{L}}(y)] = & - \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_1 \dots dy_n \left( R(g\mathcal{L}, \dots, g\mathcal{L}, W; V)(y_N, x; y) + \right. \\ & \left. - R(g\mathcal{L}, \dots, g\mathcal{L}, V; W)(y_N, y; x) \right). \end{aligned} \quad (4.20)$$

**2.2. Iterating the interaction.** Bogoliubov's formula not only defines the interacting fields it also provides for well defined time ordered products of these by multiple functional differentiation [EG73]. We have

$$\begin{aligned} T(W_1, \dots, W_n)_{g\mathcal{L}}(x_1, \dots, x_n) = & \\ = & \frac{\delta^n}{i^n \delta h_1(x_1) \dots \delta h_n(x_n)} S_{g\mathcal{L}} \left( \sum_{j=1}^n h_j W_j \right) \Big|_{h_1=\dots=h_n=0} \\ = & \sum_{m=0}^{\infty} \frac{i^m}{m!} \int dy_1 \dots dy_m \times \\ & \times R(g\mathcal{L}, \dots, g\mathcal{L}; W_1, \dots, W_n)(y_1, \dots, y_m; x_1, \dots, x_n). \end{aligned} \quad (4.21)$$

The retarded products are given by

$$R(V_1, \dots, V_m; W_1, \dots, W_n)(y_M; x_N) \sum_{I \subset M} (-)^{|I|} \overline{T}(I)(y_I) T(I^c, N)(y_{I^c}, x_N), \quad (4.22)$$

where we used our short hand notation, denoting  $N = \{1, \dots, n\}$  and  $M = \{1, \dots, m\}$ . They have retarded support, too:

$$\begin{aligned} \text{supp } R(V_1, \dots, V_m; W_1, \dots, W_n)(y_1, \dots, y_m; x_1, \dots, x_n) \subset \\ \subset \left\{ (y_1, \dots, y_m, x_1, \dots, x_n) \in \mathbb{M}^{m+n}, y_i \in \bigcup_{j=1}^n \overline{V}_-(x_j), \forall i = 1, \dots, m \right\}. \end{aligned} \quad (4.23)$$

Following [DF00a] we now study the iteration of the causal construction. We build the  $S$ -matrix of a new interaction  $\mathcal{K}_{g\mathcal{L}}$  as a formal power series in  $h \in \mathcal{D}(\mathbb{M})$ :

$$S(h\mathcal{K}_{g\mathcal{L}}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n T(h\mathcal{K}, \dots, h\mathcal{K})_{g\mathcal{L}}(x_1, \dots, x_n), \quad (4.24)$$

which is according to (4.21) and the definition of a generating functional

$$= S_{g\mathcal{L}}(h\mathcal{K}). \quad (4.25)$$

Then the corresponding relative  $S$ -matrix is given by

$$S_{h\mathcal{K}_{g\mathcal{L}}}(fW_{g\mathcal{L}}) = S(h\mathcal{K}_{g\mathcal{L}})^{-1} S(h\mathcal{K}_{g\mathcal{L}} + fW_{g\mathcal{L}}) \quad (4.26)$$

$$= S_{g\mathcal{L}}(h\mathcal{K})^{-1} S_{g\mathcal{L}}(h\mathcal{K} + fW) \quad (4.27)$$

$$= S(g\mathcal{L} + h\mathcal{K})^{-1} S(g\mathcal{L} + h\mathcal{K} + fW) \quad (4.28)$$

$$= S_{g\mathcal{L}+h\mathcal{K}}(fW). \quad (4.29)$$

This allows to define the interacting  $(W_{g\mathcal{L}})_{h\mathcal{K}_g\mathcal{L}}$ -field and  $T$ -products of them, corresponding to the field  $W_{g\mathcal{L}}$  by the Bogoliubov formula:

$$(W_{g\mathcal{L}})_{h\mathcal{K}_g\mathcal{L}}(x) = \frac{\delta}{i\delta f(x)} S_{h\mathcal{K}_g\mathcal{L}}(fW_{g\mathcal{L}}) \Big|_{f=0} \quad (4.30)$$

$$= W_{g\mathcal{L}+h\mathcal{K}}(x), \quad (4.31)$$

because of (4.29). Especially in the case  $\mathcal{K} = \mathcal{L}$  and  $h = -g$  the interacting field just returns the free field.

Now we introduce the general setting which is used in the following chapters. The interaction Lagrangian  $\mathcal{L}$  is assumed to lead to a renormalizable quantum field theory, hence  $\dim \mathcal{L} \leq 4$ . Our test function<sup>2</sup>  $g$  is assumed to be constant on an open bounded causally complete spacetime region  $\mathcal{O}$ , see figure 1.

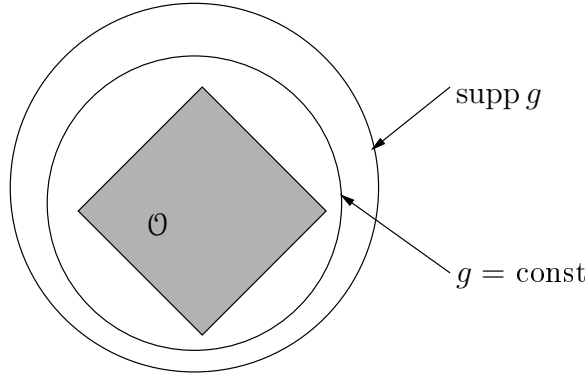


FIGURE 1. Observable algebra  $\mathfrak{A}(\mathcal{O})$ .

Then we construct the interacting fields  $W_{g\mathcal{L}}$  in that region according to

$$W_{g\mathcal{L}}(f) \forall f \in \mathcal{D}(\mathbb{M}) \text{ with } \text{supp } f \subset \mathcal{O}, W \in \mathfrak{B}. \quad (4.32)$$

The algebra of these fields is our observable algebra  $\mathfrak{A}_{g\mathcal{L}}(\mathcal{O})$ . It was shown in [BF96] that any change of the interaction outside the closure of  $\mathcal{O}$  leads to a unitary transformation of  $S_{g\mathcal{L}}(W)$  independent of  $W$  and hence of all interacting fields. Therefore the interaction fixes  $\mathfrak{A}_{g\mathcal{L}}(\mathcal{O})$  up to unitary equivalence.

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<sup>2</sup>If we also consider a sum of couplings  $g$  becomes a vector.



## CHAPTER 5

### The energy momentum tensor

In classical field theory any symmetry of the Lagrangian generates a conserved current by the Noether procedure. If the Lagrangian is not invariant but only shifts by a divergence the same procedure still applies. The current which is associated to a translation of the fields is the energy momentum tensor (EMT). If we have a localized interaction translation invariance is obviously broken and the EMT is conserved only where the localization function is constant. But this is already enough in view of our interacting observable algebras.

The Lagrangian possesses a further symmetry, namely scale invariance, if no dimensionful couplings are present. Callan, Coleman and Jackiw have shown [CCJ70] that in this situation an *improved* EMT can be defined by addition of a conserved improvement tensor. This improved tensor is also traceless. Contraction of this tensor with  $x$  defines the conserved *dilatation* current reflecting scale invariance of the Lagrangian.

We pursue another way of defining the improved tensor on the example of the massless scalar field theory: The equations of motion do not fix the Lagrangian unambiguously. We find the improved tensor as the EMT of an improved Lagrangian. A similar derivation was also given by Kasper [Kas81]. Since the improvement tensor is strictly conserved, the improved EMT is only conserved up to the breaking term of the canonical one, related to the localization of the interaction. This term also causes a breaking of the dilatation current. We discuss this classical situation in section 1.

With the classical preliminaries we study the quantum theory. In section 2 we analyze the canonical EMT for a family of theories, where the free field equation is at least of second order and the interaction contains no derivated fields. We find that exactly the same conservation equation like for the classical fields can be fulfilled if we impose a further normalization condition called *Ward identity*. The Ward identity requires a suitable normalization of  $T$ -products involving the canonical free energy momentum tensor (as a symbol  $\in \mathfrak{B}$ ). We show that the Ward identity can always be satisfied in section 3 by using the inductive method of Dütsch and Fredenhagen in [DF99]. Our result coincides with a similar result that was derived for the energy momentum tensor in the framework of Zimmermann's normal product quantization [Zim73a] by Lowenstein [Low71] and also by Zimmermann [Zim84]. We show that the interacting momentum operator (as the corresponding charge) implements the right commutation relation with the interacting fields.

In their paper [CJ71] Coleman and Jackiw argued that the trace of the improved energy momentum tensor generates an anomaly in the perturbative

interacting quantum theory. Lowenstein has verified this statement for Zimmermann's normal products in [Low71]. Later, Zimmermann has given a derivation of this anomaly in [Zim84]. He verified a conjecture by Minkowski [Min76] that it can be normalized to be proportional to the  $\beta$ -function of the Callan-Symanzik equation. This statement already contains a result of Schroer [Sch71] that the anomaly vanishes if the coupling is a zero of the  $\beta$ -function. A more comprehensive result, also covering possible conformal anomalies, was given by Kraus and Sibold [KS92, KS93] using the framework of algebraic renormalization.

In accordance to these results we show in section 5 that a conserved (up to the expected  $\partial g$  breaking) improved EMT inevitably leads to the trace anomaly in  $\varphi^4$ -theory. The definition of the improvement tensor requires a new relation between interacting fields induced by a corresponding Ward identity which is proved in section 6. The simultaneous validity of both Ward identities forces the trace anomaly to be present. The EMT and its improved counterpart both define the same momentum operator. The breaking of dilatations given by the trace leads to anomalous contributions to the dimension of the interacting fields described in section 7. We mention that dilatations were also quite recently studied in local perturbation theory by Grigore [Gri00]. But his main focus is on the  $S$ -matrix whereas we focus on the interacting fields.

### 1. The energy momentum tensor in classical field theory

We discuss the EMT in a classical field theory. The Lagrangian depends on the fields  $\phi_j^{\text{class}}, j = 1, \dots, r$  and their first and second derivatives. We assume that it also depends on  $x$  explicitly via a coupling term  $-g\mathcal{L}_{\text{int}}^{\text{class}}$  which is assumed to contain no derivated fields:  $\mathcal{L}^{\text{class}} = \mathcal{L}^{\text{class}}(\phi_l^{\text{class}}, \phi_{l,\mu}^{\text{class}}, \phi_{l,\mu\nu}^{\text{class}}, x)$ . Then the Euler-Lagrange equations read:

$$\partial_\mu \partial_\nu \frac{\partial \mathcal{L}^{\text{class}}}{\partial \phi_{l,\mu\nu}^{\text{class}}} - \partial_\mu \frac{\partial \mathcal{L}^{\text{class}}}{\partial \phi_{l,\mu}^{\text{class}}} + \frac{\partial \mathcal{L}^{\text{class}}}{\partial \phi_l^{\text{class}}} = 0. \quad (5.1)$$

The EMT is the current associated to a spacetime translation of the fields:  $\phi^{\text{class}}(x) \rightarrow \phi^{\text{class}}(x+a)$ . By the Noether procedure we find the EMT to be:

$$\begin{aligned} \Theta^{\text{class} \mu\nu} &= \frac{\partial \mathcal{L}^{\text{class}}}{\partial \phi_{l,\mu}^{\text{class}}} \partial^\nu \phi_l^{\text{class}} - \left( \partial_\rho \frac{\partial \mathcal{L}^{\text{class}}}{\partial \phi_{l,\mu\rho}^{\text{class}}} \right) \partial^\nu \phi_l^{\text{class}} + \\ &+ \frac{\partial \mathcal{L}^{\text{class}}}{\partial \phi_{l,\mu\rho}^{\text{class}}} \partial_\rho \partial^\nu \phi_l^{\text{class}} - \eta^{\mu\nu} \mathcal{L}^{\text{class}}. \end{aligned} \quad (5.2)$$

In this situation the “conservation” equation reads:

$$\partial_\mu \Theta^{\text{class} \mu\nu} = \partial^\nu g \mathcal{L}_{\text{int}}^{\text{class}}. \quad (5.3)$$

In the following we investigate two specific models:

**1.1. The general first order model.** In this subsection we restrict ourselves to the case that there are no twice derivated fields present. The free Lagrangian is quadratic in the fields  $\phi_j^{\text{class}}, \phi_{j,\mu}^{\text{class}}, j = 1, \dots, r$ . The interaction

is given by the term above (containing no derivated fields).

$$\mathcal{L}^{\text{class}} = \mathcal{L}_0^{\text{class}} - g\mathcal{L}_{\text{int}}^{\text{class}} \quad (5.4)$$

$$\mathcal{L}_0^{\text{class}} = \frac{1}{2}K_{jl}^{\mu\nu}\partial_\mu\phi_j^{\text{class}}\partial_\nu\phi_l^{\text{class}} + \frac{1}{2}G_{jl}^\mu\partial_\mu\phi_j^{\text{class}}\phi_l^{\text{class}} - \frac{1}{2}M_{jl}\phi_j^{\text{class}}\phi_l^{\text{class}} \quad (5.5)$$

with  $K_{jl}^{\mu\nu}, G_{jl}^\mu, M_{lj} \in \mathbb{C}$ . The  $K, G$  and  $M$  are supposed to possess the following symmetries:  $K_{jl}^{\mu\nu} = K_{lj}^{\nu\mu} = K_{lj}^{\mu\nu}, G_{jl}^\mu = -G_{lj}^\mu$  and  $M_{jl} = M_{lj}$ . The Euler-Lagrange equations read

$$(K_{jl}^{\mu\nu}\partial_\mu\partial_\nu + G_{jl}^\mu\partial_\mu + M_{jl})\phi_l^{\text{class}} \doteq D_{jl}\phi_l^{\text{class}} = -g\frac{\partial\mathcal{L}_{\text{int}}^{\text{class}}}{\partial\phi_j^{\text{class}}}. \quad (5.6)$$

The EMT defined by (5.2) is called the *canonical* EMT. It is given by

$$\Theta_{\text{can}}^{\text{class}\mu\nu} = \Theta_{0\text{can}}^{\text{class}\mu\nu} - \eta^{\mu\nu}g\mathcal{L}_{\text{int}}^{\text{class}} \quad (5.7)$$

$$\begin{aligned} \Theta_{0\text{can}}^{\text{class}\mu\nu} = & K_{lk}^{\mu\rho}\partial_\rho\phi_l^{\text{class}}\partial^\nu\phi_k^{\text{class}} + \frac{1}{2}G_{lk}^\mu\partial^\nu\phi_l^{\text{class}}\phi_k^{\text{class}} + \\ & - \frac{1}{2}\eta^{\mu\nu}\left(K_{lk}^{\rho\sigma}\partial_\rho\phi_l^{\text{class}}\partial_\sigma\phi_k^{\text{class}} + G_{lk}^\rho\partial_\rho\phi_l^{\text{class}}\phi_k^{\text{class}} - M_{lk}\phi_l^{\text{class}}\phi_k^{\text{class}}\right). \end{aligned} \quad (5.8)$$

In section 2 we show that the “conservation” equation (5.3) can also be fulfilled in the interacting quantum field theory.

**1.2. The massless  $(\phi^{\text{class}})^4$ -theory.** If no dimensionful couplings are present, it is always possible to construct a conserved and traceless EMT. This tensor is called the *improved* EMT and was first introduced by Callan, et. al. in [CCJ70]. We derive this tensor in  $(\phi^{\text{class}})^4$ -theory as the EMT of an improved Lagrangian making use of the ambiguity in the definition of the Lagrangian. A derivation of this kind was already performed by Kasper [Kas81].

The equations of motion read:

$$\square\phi^{\text{class}} = -g\frac{\partial\mathcal{L}_{\text{int}}^{\text{class}}}{\partial\phi^{\text{class}}}. \quad (5.9)$$

Since a total derivative in the Lagrangian does not change the equations of motion they originate from both following expressions:

$$\mathcal{L}_{\text{can}}^{\text{class}} = \frac{1}{2}\partial_\rho\phi^{\text{class}}\partial^\rho\phi^{\text{class}} - g\mathcal{L}_{\text{int}}^{\text{class}}, \quad (5.10)$$

$$\mathcal{L}_{\text{imp}}^{\text{class}} = \frac{1}{6}\partial_\rho\phi^{\text{class}}\partial^\rho\phi^{\text{class}} - \frac{1}{3}\phi^{\text{class}}\square\phi^{\text{class}} - g\mathcal{L}_{\text{int}}^{\text{class}}. \quad (5.11)$$

The corresponding EMT's are called the canonical and the improved one, respectively. The first one already follows from the last subsection:

$$\Theta_{\text{can}}^{\text{class } \mu\nu} = \partial^\mu \phi^{\text{class}} \partial^\nu \phi^{\text{class}} - \frac{1}{2} \eta^{\mu\nu} \partial_\rho \phi^{\text{class}} \partial^\rho \phi^{\text{class}} + g \eta^{\mu\nu} \mathcal{L}_{\text{int}}^{\text{class}}, \quad (5.12)$$

$$\Theta_{\text{imp}}^{\text{class } \mu\nu} = \frac{2}{3} \partial^\mu \phi^{\text{class}} \partial^\nu \phi^{\text{class}} - \frac{1}{3} \phi^{\text{class}} \partial^\mu \partial^\nu \phi^{\text{class}} + \quad (5.13)$$

$$- \frac{1}{6} \eta^{\mu\nu} \partial_\rho \phi^{\text{class}} \partial^\rho \phi^{\text{class}} + \frac{1}{12} \eta^{\mu\nu} \phi^{\text{class}} \square \phi^{\text{class}} \quad (5.14)$$

$$= \Theta_{\text{can}}^{\text{class } \mu\nu} - \frac{1}{3} I^{\text{class } \mu\nu}, \quad (5.15)$$

where we have introduced the conserved improvement tensor

$$I^{\text{class } \mu\nu} = \partial^\mu \phi^{\text{class}} \partial^\nu \phi^{\text{class}} + \phi^{\text{class}} \partial^\mu \partial^\nu \phi^{\text{class}} + \quad (5.16)$$

$$- \eta^{\mu\nu} \partial_\rho \phi^{\text{class}} \partial^\rho \phi^{\text{class}} - \eta^{\mu\nu} \phi^{\text{class}} \square \phi^{\text{class}} \quad (5.17)$$

Contracting the indices, we find  $\eta_{\mu\nu} \Theta_{\text{imp}}^{\text{class } \mu\nu} = 0$ . The improved tensor gives rise to the dilatation current:

$$D^{\text{class } \mu} \doteq x_\nu \Theta_{\text{imp}}^{\text{class } \mu\nu}. \quad (5.18)$$

Its conservation equation reads:

$$\partial_\mu D^{\text{class } \mu} = \eta_{\mu\nu} \Theta_{\text{imp}}^{\text{class } \mu\nu} + x_\nu \partial_\mu \Theta_{\text{imp}}^{\text{class } \mu\nu} = x^\mu \partial_\mu g \mathcal{L}_{\text{int}}^{\text{class}}. \quad (5.19)$$

The dilatation current is the Noether current corresponding to the scaling  $\phi^{\text{class}}(x) \rightarrow e^{d\alpha} \phi^{\text{class}}(e^\alpha x)$  with  $d = 1$  of the improved Lagrangian (5.11). On the other hand we can derive the dilatations from the canonical Lagrangian (5.10). In this case we find:

$$\tilde{D}^{\text{class } \mu} = x_\nu \Theta_{\text{can}}^{\text{class } \mu\nu} + \phi^{\text{class}} \partial^\mu \phi^{\text{class}}, \quad (5.20)$$

with the same conservation equation. The zero component of the difference is a divergence w.r.t. the space coordinates and therefore does not contribute to the charge:

$$\tilde{D}^{\text{class } 0} - D^{\text{class } 0} = \frac{1}{3} \partial_j (x^{[j} \partial^{0]}) (\phi^{\text{class}})^2. \quad (5.21)$$

In section 5 the corresponding QFT is considered.

## 2. The canonical quantum energy momentum tensor

We now discuss the perturbative quantum fields which are associated to the canonical EMT. While for the free theory all equations from classical field theory can be carried over due to the Wick ordering procedure, the interacting quantum fields require a special normalization which is implied by a Ward identity. This identity enables to conserve the classical structure also in the quantum theory.



**2.1. Free quantum theory.** The classical fields from the last section may now serve as the symbols from our auxiliary variable algebra  $\mathfrak{B}$ . For our corresponding quantum fields  $\varphi_j$  we assume the same equations of motion to be satisfied

$$D_{jl}T(\varphi_l) = 0. \quad (5.22)$$

By investigation of the  $K, G$  and  $M$  we find that this covers a lot of equations like Klein-Gordon and Dirac equation. The commutator

$$[T(\varphi_j)(x), T(\varphi_k)(y)] = i\Delta_{jk}(x - y), \quad (5.23)$$

may therefore be an anticommutator in the case of Fermi fields. Now we regard  $\Theta_{0\text{can}}^{\mu\nu} \in \mathfrak{B}$  from (5.8) as a symbol ( $\phi^{\text{class}} \rightarrow \varphi$ ):

$$\begin{aligned} \Theta_{0\text{can}}^{\mu\nu} = & K_{lk}^{\mu\rho} \partial_\rho \varphi_l \partial^\nu \varphi_k + \frac{1}{2} G_{lk}^\mu \partial^\nu \varphi_l \varphi_k + \\ & - \frac{1}{2} \eta^{\mu\nu} (K_{lk}^{\rho\sigma} \partial_\rho \varphi_l \partial_\sigma \varphi_k + G_{lk}^\rho \partial_\rho \varphi_l \varphi_k - M_{lk} \varphi_l \varphi_k). \end{aligned} \quad (5.24)$$

Then  $T(\Theta_{0\text{can}}^{\mu\nu}) =: \Theta_{0\text{can}}^{\mu\nu}$  defines the free canonical quantum EMT. Since the equations of motion hold inside the Wick ordering we maintain the conservation

$$\partial_\mu : \Theta_{0\text{can}}^{\mu\nu} := 0. \quad (5.25)$$

**2.2. Interacting quantum theory.** Applying the framework of perturbative interacting fields introduced in the last chapter we investigate the consequences of switching on an interaction. We assume that the interaction  $\mathcal{L}$  contains no derivated fields. Corresponding to the free canonical EMT we construct the interacting counterpart. According to (4.10) we have:

$$\Theta_{0\text{can}g\mathcal{L}}^{\mu\nu}(x) \doteq \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_1 \dots dy_n R(g\mathcal{L}, \dots, g\mathcal{L}; \Theta_{0\text{can}}^{\mu\nu})(y_1, \dots, y_n; x). \quad (5.26)$$

But this is only the part corresponding to the free fields. The total tensor  $\Theta_{\text{can}g\mathcal{L}}^{\mu\nu}$  receives another contribution from the interaction (cp. (5.7)):

$$\Theta_{\text{can}g\mathcal{L}}^{\mu\nu} \doteq \Theta_{0\text{can}g\mathcal{L}}^{\mu\nu} + \eta^{\mu\nu} g \mathcal{L}_{g\mathcal{L}}. \quad (5.27)$$

Since  $g$  is of compact support global translation invariance is broken. Hence we expect the conservation equation to be satisfied that takes account of the non-invariance of the coupling function (cp. (5.3)):

$$\partial_\mu \Theta_{\text{can}g\mathcal{L}}^{\mu\nu} = \partial^\nu g \mathcal{L}_{g\mathcal{L}}. \quad (5.28)$$

If this equation is true, we have local conservation on  $\mathcal{O}$ :<sup>1</sup>

$$\Theta_{\text{can}g\mathcal{L}}^{\mu\nu}(\partial_\mu f) = 0, \quad \forall f \text{ with } \text{supp } f \subset \mathcal{O}. \quad (5.29)$$

Equation (5.28) is the main statement. We now give a formulation in terms of the perturbative contributions. It comes out that the conservation can be completely discussed on the level of  $T$ -products. The corresponding equation is a Ward identity involving the free canonical EMT.

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<sup>1</sup>We have assumed  $g|_{\mathcal{O}} = \text{const} \Rightarrow \partial_\mu g|_{\mathcal{O}} = 0$ .

Inserting the definition of  $\Theta_{\text{can } g\mathcal{L}}^{\mu\nu}$  from (5.27) we see that (5.28) is equivalent to

$$\partial_\mu \Theta_{0\text{can } g\mathcal{L}}^{\mu\nu} = -g \partial^\nu \mathcal{L}_{g\mathcal{L}}. \quad (5.30)$$

We expand this into the formal power series in the coupling. The RHS becomes

$$\begin{aligned} -g \partial^\nu \mathcal{L}_{g\mathcal{L}}(x) &= \\ &= -g(x) \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_1 \dots dy_n \partial_x^\nu R(\mathcal{L}, \dots, \mathcal{L}; \mathcal{L})(y_1, \dots, y_n; x) g(y_1) \dots g(y_n) \\ &= i \sum_{n=0}^{\infty} \frac{i^{n+1}}{n!} \int dy_1 \dots dy_{n+1} g(y_1) \dots g(y_{n+1}) \times \\ &\quad \times \frac{1}{n+1} \sum_{k=1}^{n+1} \partial_x^\nu R(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}, \dots, \mathcal{L}; \mathcal{L})(y_1, \dots, \mathcal{L}, \dots, y_{n+1}; x) \delta(y_k - x) \\ &= i \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dy_1 \dots dy_n g(y_1) \dots g(y_n) \times \\ &\quad \times \sum_{k=1}^n \partial_x^\nu R(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}, \dots, \mathcal{L}; \mathcal{L})(y_1, \dots, \mathcal{L}, \dots, y_n; x) \delta(y_k - x). \end{aligned} \quad (5.31)$$

The expansion of  $\Theta_{0\text{can } g\mathcal{L}}^{\mu\nu}$  was already given at the beginning. Then (5.30) is fulfilled if

$$\begin{aligned} \partial_\mu^x R(\mathcal{L}, \dots, \mathcal{L}; \Theta_{0\text{can}}^{\mu\nu})(y_1, \dots, y_n; x) &= \\ &= i \sum_{k=1}^n \partial_x^\nu R(\mathcal{L}, \dots, \mathcal{L}, \mathcal{L}, \dots, \mathcal{L}; \mathcal{L})(y_1, \dots, \mathcal{L}, \dots, y_n; x) \delta(y_k - x) \end{aligned} \quad (5.32)$$

is satisfied to all orders. The  $R$ -products are completely determined in terms of the  $T$ -products (4.12). We show that it is sufficient to prove the following Ward identity:<sup>2</sup>

$$\begin{aligned} \partial_\mu^x T(W_1, \dots, W_n, \Theta_{0\text{can}}^{\mu\nu})(y_1, \dots, y_n, x) &= \\ &= i \sum_{k=1}^n \delta(y_k - x) \partial_k^\nu T(W_1, \dots, W_n)(y_1, \dots, y_n), \end{aligned} \quad (\text{WI } 1)$$

for all  $W_i \in \mathfrak{B}$  which are (not necessarily proper) sub monomials of the coupling  $\mathcal{L}$ . We therefore prove the statement for all  $W_i \in \mathfrak{B}$  that contain no derivated fields and have  $\dim \leq 4$ .

---

<sup>2</sup>The index on the derivative on the RHS refers to the respective  $y$ -coordinate.

The Ward identity **WI 1** can be integrated to the functional equation:

$$\begin{aligned} \partial_\mu^x \frac{\delta}{i\delta f_{\mu\nu}(x)} S\left(g\mathcal{L} + f \cdot \Theta_{0\text{can}} + \sum_{j=1}^s h_j W_j\right) \Big|_{f=0} &= \\ &= -\left(g(x) \partial_x^\nu \frac{\delta}{i\delta g(x)} + \sum_{j=1}^s h_j(x) \partial_x^\nu \frac{\delta}{i\delta h_j(x)}\right) S\left(g\mathcal{L} + \sum_{j=1}^s h_j W_j\right). \end{aligned} \quad (5.33)$$

Multiplying with  $S(g\mathcal{L})^{-1}$  from the left and expanding in powers of the coupling yields (5.32). Using

$$\frac{\delta}{i\delta f} S(g\mathcal{L} + f \cdot \Theta_{0\text{can}})^{-1} \Big|_{f=0} = -S(g\mathcal{L})^{-1} \frac{\delta}{i\delta f} S(g\mathcal{L} + f \cdot \Theta_{0\text{can}}) \Big|_{f=0} S(g\mathcal{L})^{-1} \quad (5.34)$$

we find that (5.33) also holds for the inverse functional. This implies the corresponding Ward identity for the  $\overline{T}$ -products to have a minus sign. A simple calculation also shows that (5.33) implies

$$\begin{aligned} \partial_\mu^x \frac{\delta}{i\delta f_{\mu\nu}(x)} S_{g\mathcal{L}+f\cdot\Theta_{0\text{can}}}(hW) \Big|_{f=0} &= \\ &= -\left(g(x) \partial_x^\nu \frac{\delta}{i\delta g(x)} + h(x) \partial_x^\nu \frac{\delta}{i\delta h(x)}\right) S_{g\mathcal{L}}(hW). \end{aligned} \quad (5.35)$$

Expanding this equation in  $n$ 'th order  $g$  and first order  $h$  results in the following identity for the  $R$ -products:

$$\begin{aligned} \partial_\mu^{x_1} R(N, \Theta_{0\text{can}}^{\mu\nu}; W)(y_N, x_1; x_2) &= i \sum_{k \in N} \delta(y_k - x_1) \partial_k^\nu R(N; W)(y_N; x_2) + \\ &+ i\delta(x_2 - x_1) \partial_{x_2}^\nu R(N; W)(y_N; x_2). \end{aligned} \quad (5.36)$$

The next section gives a proof of **WI 1**.

### 3. Proof of the Ward identity

Dütsch and Fredenhagen have presented a very general framework for proving a Ward identity of this kind in [DF99]. It was generalized by Boas [Boa99] in the presence of derivated fields. This is our situation here and we apply their methods. The strategy is as follows: A possible violation of the Ward identity is called an *anomaly*. Since all  $T$ -products are supposed to fulfil the normalization conditions **N0** – **N4** we perform a double induction, one over  $n$  and one over the degree (number of generators) of the Wick monomials. We assume that the anomaly in both lower orders is zero.

- **Step 1.** The commutator of the anomaly with the free fields vanishes. Therefore it can appear only in the vacuum sector.
- **Step 2.** If one  $W_i$  is a generator (which is the lowest degree sub monomial), the anomaly vanishes due to **N4**.
- **Step 3.** Because of **N1** and the induction on  $n$ , the anomaly is a Poincaré covariant  $\mathbb{C}$ -number distribution with support on the total diagonal. We

show that it vanishes by an appropriate normalization, i.e. adding a  $\delta$ -polynomial with the right symmetry properties (**N0**).<sup>3</sup>

To save some space we use the short hand notations from above. We define the anomaly  $a$  by:

$$a^\nu(x, y_N) \doteq \partial_\mu^x T(\Theta_{0\text{can}}^{\mu\nu}, N)(x, y_N) - i \sum_{k=1}^n \delta(y_k - x) \partial_k^\nu T(N)(y_N). \quad (5.37)$$

**Step 1.** We commute the anomaly with the free fields.<sup>4</sup> We use a double induction, one on  $n$  and the other on the degree of the Wick sub monomials. Using (**N3**) we need the sub monomials of  $\Theta_{0\text{can}}^{\mu\nu}$ :

$$\frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_j} = -\frac{1}{2} G_{jl}^\mu \partial^\nu \varphi_l + \frac{1}{2} \eta^{\mu\nu} G_{jl}^\rho \partial_\rho \varphi_l + \eta^{\mu\nu} M_{jl} \varphi_l \quad (5.38)$$

$$\frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_{j,\rho}}, = K_{jl}^{\mu\rho} \partial^\nu \varphi_l + 2\eta^{\nu(\rho} K_{jl}^{\mu)\sigma} \partial_\sigma \varphi_l + \eta^{\nu(\rho} G_{jl}^{\mu)} \varphi_l. \quad (5.39)$$

We explicitly distinguished between the basic generators and the first order ones. Here and in the following the sums only run over the basic generators therefore. Equations (5.38),(5.39) are linear in the fields. We demand  $T$ -products containing once derivated basic generators to fulfil the following normalization:<sup>5</sup>

$$\partial_\mu^x T(\varphi_j, N)(x, y_N) = T(\partial_\mu \varphi_j, N)(x, y_N). \quad (5.40)$$

This translates into

$$\partial_\mu \varphi_{jg\mathcal{L}}(x) = (\partial_\mu \varphi_j)_{g\mathcal{L}} \quad (5.41)$$

---

<sup>3</sup>If the current itself is a sub monomial of the coupling, this may lead to non trivial conditions like in [DF99].

<sup>4</sup>We use the symbols  $\varphi_{l,\mu}$  and  $\partial_\mu \varphi_l$  synonymously.

<sup>5</sup>Because of (3.60) this means  $\omega_0(T(\varphi_{j,\mu}, \varphi_k)(x, y)) = \partial_\mu^x \omega_0(T(\varphi_j, \varphi_k)(x, y))$ . If the fields  $\varphi_j, \varphi_k$  are bosonic with mass dimension 1 this is automatically fulfilled because of the negative singular order. In the case of two Fermi fields with mass dimension  $\frac{3}{2}$  and non vanishing anticommutator, e.g.  $\psi, \bar{\psi}$  we have:  $\omega_0(T(\psi_{j,\mu}, \bar{\psi})(x, y)) = i\partial_\mu^x S^F(x - y) + c\gamma_\mu \delta(x - y)$ . The normalization (5.40) requires  $c = 0$ .

for the interacting fields. We calculate the following relevant term:

$$\begin{aligned}
& \partial_\mu^x \left\{ T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_j}, N \right) (x, y_N) \Delta_{ji}(x-z) + T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_{j,\rho}}, N \right) (x, y_N) \partial_\rho \Delta_{ji}(x-z) \right\} = \\
& = \partial_\mu^x \left\{ \Delta_{ji}(x-z) \left[ -\frac{1}{2} G_{jl}^\mu \partial_x^\nu + \frac{1}{2} \eta^{\mu\nu} G_{jl}^\rho \partial_\rho^x + \eta^{\mu\nu} M_{jl} \right] + \right. \\
& \quad \left. + \partial_\rho \Delta_{ji}(x-z) \left[ K_{jl}^{\mu\rho} \partial_x^\nu + \eta^{\nu\rho} K_{jl}^{\mu\sigma} \partial_\sigma^x - \eta^{\mu\nu} K_{jl}^{\rho\sigma} \partial_\sigma^x + \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \eta^{\nu\rho} G_{jl}^\mu - \frac{1}{2} \eta^{\mu\nu} G_{jl}^\rho \right] \right\} T(\varphi_l, N) (x, y_N) \\
& = \left( -\partial_\mu \Delta_{ji}(x-z) G_{jl}^\mu \partial_x^\nu + M_{jl} \Delta_{ji}(x-z) \partial_x^\nu + \right. \\
& \quad \left. + \partial_\mu \partial_\rho \Delta_{ji}(x-z) K_{jl}^{\mu\rho} \partial_x^\nu + \partial^\nu \Delta_{ji}(x-z) M_{jl} + \right. \\
& \quad \left. + \partial^\nu \Delta_{ji}(x-z) G_{jl}^\mu \partial_\mu^x + \partial^\nu \Delta_{ji}(x-z) K_{jl}^{\mu\sigma} \partial_\mu^x \partial_\sigma^x \right) T(\varphi_l, N) (x, y_N) \\
& = (D_{lj} \Delta_{ji}(x-z) \partial_x^\nu + \partial^\nu \Delta_{ji}(x-z) D_{jl}^x) T(\varphi_l, N) (x, y_N) \\
& = \partial^\nu \Delta_{ji}(x-z) D_{jl}^x T(\varphi_l, N) (x, y_N),
\end{aligned} \tag{5.42}$$

since  $\Delta$  is a solution of the free field equation. Now we commute the anomaly with the free basic fields and use **N3**:

$$\begin{aligned}
& [a^\nu(x, y_N), \varphi_i(z)] = \\
& = \partial_\mu^x [T(\Theta_{0\text{can}}^{\mu\nu}, N) (x, y_N), \varphi_i(z)] + \\
& \quad - i \sum_{k=1}^n \delta(y_k - x) \partial_k^\nu [T(N) (y_N), \varphi_i(z)] \\
& = i \partial_\mu^x \left\{ T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_j}, N \right) (x, y_N) \Delta_{ji}(x-z) + \right. \\
& \quad \left. + T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_{j,\rho}}, N \right) (x, y_N) \partial_\rho \Delta_{ji}(x-z) \right\} + \\
& \quad + i \sum_{l=1}^n \partial_\mu^x T \left( \Theta_{0\text{can}}^{\mu\nu}, W_1, \dots, \frac{\partial W_l}{\partial \varphi_j}, \dots, W_n \right) (x, y_N) \Delta_{ji}(y_l - z) + \\
& \quad + \sum_{k=1}^n \delta(y_k - x) \partial_k^\nu \left\{ \sum_{l=1}^n T \left( W_1, \dots, \frac{\partial W_l}{\partial \varphi_j}, \dots, W_n \right) (y_N) \Delta_{ji}(y_l - z) \right\}.
\end{aligned} \tag{5.43}$$

If the derivative on the last line acts on  $T$  it cancels the third line by the induction hypothesis. Only the term with  $l = k$  and the derivative on  $\Delta$  remains.

The  $\delta$  function allows to put  $y_k = x$  and by inserting (5.42) we end up with

$$= \partial^\nu \Delta_{ji}(x - z) \left[ i D_{jl}^x T(\varphi_l, N)(x, y_N) + \right. \quad (5.44)$$

$$\left. + \sum_{k=1}^n \delta(y_k - x) T \left( W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) \right] \quad (5.45)$$

$$= 0, \quad (5.46)$$

because of (N4). Since the Ward identity commutes with all free fields the anomaly is a  $\mathbb{C}$ -number. Because of the causal Wick expansion (3.55) it can only appear in the vacuum sector:

$$a^\nu(x, y_N) = \langle \Omega, a^\nu(x, y_N) \Omega \rangle. \quad (5.47)$$

**Step 2.** We prove that the Ward identity is compatible with the normalization condition (N4). Using the equivalent formulation (3.60)

$$\begin{aligned} \langle \Omega, T(N, \varphi_i)(y_N, z) \Omega \rangle &= \\ &= i \sum_{k=1}^n \Delta_{ij}^F(z - y_k) \left\langle \Omega, T \left( W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) (y_N) \Omega \right\rangle + \\ &\quad - i \sum_{k=1}^n \partial_\rho \Delta_{ij}^F(z - y_k) \left\langle \Omega, T \left( W_1, \dots, \frac{\partial W_k}{\partial \varphi_{j,\rho}}, \dots, W_n \right) (y_N) \Omega \right\rangle. \end{aligned} \quad (5.48)$$

with  $\Delta_{ij}^F(y - x) = \langle \Omega, \varphi_i(y) \varphi_j(x) \Omega \rangle$ . Then a calculation along the lines of (5.42) shows

$$\begin{aligned} \partial_\mu^x \left\{ \Delta_{ij}^F(z - x) \left\langle \Omega, T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_j}, N \right) (x, y_N) \Omega \right\rangle + \right. \\ \left. - \partial_\rho \Delta_{ij}^F(z - x) \left\langle \Omega, T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_{j,\rho}}, N \right) (x, y_N) \Omega \right\rangle \right\} = \\ = (D_{jl} \Delta_{ij}^F(z - x) \partial_x^\nu - \partial^\nu \Delta_{ij}^F(z - x) D_{jl}^x) \langle \Omega, T(\varphi_l, N)(x, y_N) \Omega \rangle \\ = (\delta_{li} \delta(z - x) \partial_x^\nu - \partial^\nu \Delta_{ij}^F(z - x) D_{jl}^x) \langle \Omega, T(\varphi_l, N)(x, y_N) \Omega \rangle, \end{aligned} \quad (5.49)$$

since  $\Delta^F$  is a Green's function of the equation of motion:  $D_{jl} \Delta_{ij}^F = D_{lj} \Delta_{ji}^F = \delta_{li} \delta$ . We set  $a^\nu(x, y_N, z)$  like before with  $W_{n+1} \doteq \varphi_i$  and compute its vacuum

expectation value. We obtain

$$\begin{aligned}
a^\nu(x, y_N, z) &= \\
&= i\partial_\mu^x \left\{ \Delta_{ij}^F(z-x) \left\langle \Omega, T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_j}, N \right) (x, y_N) \Omega \right\rangle + \right. \\
&\quad \left. - \partial_\rho \Delta_{ij}^F(z-x) \left\langle \Omega, T \left( \frac{\partial \Theta_{0\text{can}}^{\mu\nu}}{\partial \varphi_{j,\rho}}, N \right) (x, y_N) \Omega \right\rangle \right\} + \\
&\quad + i \sum_{l=1}^n \Delta_{ij}^F(z-y_l) \partial_\mu^x \left\langle \Omega, T \left( \Theta_{0\text{can}}^{\mu\nu}, W_1, \dots, \frac{\partial W_l}{\partial \varphi_j}, \dots, W_n \right) (x, y_N) \Omega \right\rangle + \\
&\quad - i\delta(z-x) \partial_z^\nu \langle \Omega, T(\varphi_i, N)(z, y_N) \Omega \rangle + \\
&\quad + \sum_{k=0}^n \delta(y_k - x) \partial_k^\nu \left\{ \sum_{l=1}^n \Delta_{ij}^F(z-y_l) \left\langle \Omega, T \left( W_1, \dots, \frac{\partial W_l}{\partial \varphi_j}, \dots, W_n \right) (y_N) \Omega \right\rangle \right\}.
\end{aligned} \tag{5.50}$$

Again, if in the last line the derivative acts on the  $T$ -products these terms cancel the third line by the induction hypothesis. The term  $l = k$  with the derivative on  $\Delta^F$  remains. Inserting (5.49) gives:

$$= -i\partial^\nu \Delta_{ij}^F(z-x) D_{jl}^x \langle \Omega, T(\varphi_l, N)(x, y_N) \Omega \rangle + \tag{5.51}$$

$$\begin{aligned}
&\quad - \sum_{k=0}^n \delta(y_k - x) \partial^\nu \Delta_{ij}^F(z-y_k) \left\langle \Omega, T \left( W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n \right) (y_N) \Omega \right\rangle \\
&= 0,
\end{aligned} \tag{5.52}$$

because of (N4).

**Step 3.** We show how the anomaly can be removed by an appropriate normalization. The above steps have shown that it has the following form:

$$a^\nu(x, y_N) = \partial_\mu^x \langle \Omega, T(\Theta_{0\text{can}}^{\mu\nu}, N)(x, y_N) \Omega \rangle - i \sum_{k=1}^n \delta(y_k - x) \partial_k^\nu \langle \Omega, T(N)(y_N) \Omega \rangle \tag{5.53}$$

$$= M^\nu(\partial) \delta(y_1 - x) \dots \delta(y_n - x). \tag{5.54}$$

Here,  $\partial = (\partial_1, \dots, \partial_n)$  and  $M^\nu$  is a Lorentz vector valued polynomial of degree  $\leq 5$ , since  $\text{sing ord} \langle \Omega, T(\Theta_{0\text{can}}^{\mu\nu}, N), \Omega \rangle = \dim \Theta_{0\text{can}}^{\mu\nu} + \sum_{i=1}^n \dim W_i - 4n \leq 4$  according to (3.61). If  $M^\nu(\partial)$  has the form

$$M^\nu(\partial) = \sum_{i=1}^n \partial_\mu^i M_1^{\mu\nu}(\partial), \tag{5.55}$$

with  $M_1$  again a polynomial, the normalization  $T(\Theta_{0\text{can}}^{\mu\nu}, N) \rightarrow T(\Theta_{0\text{can}}^{\mu\nu}, N) + M_1^{\mu\nu}(\partial)\delta$  removes the anomaly.

We show that this is the case. We introduce the free momentum operator:<sup>6</sup>

$$P^\mu \doteq \int d^3\mathbf{x} \Theta_{0\text{can}}^{0\nu}(x). \quad (5.56)$$

It is a hermitian operator that annihilates the vacuum.

For every  $(y_1, \dots, y_n)$  we take a double cone  $\mathcal{O}$  with  $y_i \in \mathcal{O}$  for all  $i = 1, \dots, n$ . Choosing a  $g$ , with  $g|_{\overline{\mathcal{O}}} = 1$  we decompose  $\partial_\mu g = a_\mu - b_\mu$ , such that  $\text{supp } a_\mu \cap (\overline{\mathcal{V}}_- + \mathcal{O}) = \text{supp } b_\mu \cap (\overline{\mathcal{V}}_+ + \mathcal{O}) = \emptyset$ . We smear out the first term on the r.h.s of (5.37) with this  $g$  and use the causal factorization of the  $T$ -products:

$$\int dx \partial_\mu^x T(\Theta_{0\text{can}}^{\mu\nu}, N)(x, y_N) g(x) = \quad (5.57)$$

$$= -\Theta_{0\text{can}}^{\mu\nu}(a_\mu) T(N)(y_N) + T(N)(y_N) \Theta_{0\text{can}}^{\mu\nu}(b_\mu) \quad (5.58)$$

$$= -[\Theta_{0\text{can}}^{\mu\nu}(a_\mu), T(N)(y_N)] - T(N)(y_N) \Theta_{0\text{can}}^{\mu\nu}(\partial_\mu g).$$

The second term vanishes because  $\Theta_{0\text{can}}^{\mu\nu}$  is a conserved current. Then, in the commutator  $\Theta_{0\text{can}}^{\mu\nu}(a_\mu)$  can be replaced by  $-P^\nu$ , since  $T(N)$  is localized in  $\mathcal{O}$ :

$$= [P^\nu, T(N)(y_N)]. \quad (5.59)$$

Therefore the vacuum expectation value of (5.59) vanishes. Smearing the second term of (5.37) with  $g$  and taking the vacuum expectation value we find:

$$i \int dx \sum_{j=1}^n \delta(y_j - x) \partial_j^\nu \langle \Omega, T(N)(y_N) \Omega \rangle g(x) = i \sum_{j=1}^n \partial_j^\nu \langle \Omega, T(N)(y_N) \Omega \rangle = 0, \quad (5.60)$$

because of translation invariance. Hence we get<sup>7</sup>

$$\int dx a^\nu(x, y_N) = 0. \quad (5.61)$$

To prove (5.55) we work in Fourier space:

$$\widehat{a^\nu}(x, p_1, \dots, p_n) = \int dy_1 \dots dy_n a^\nu(x, y_1, \dots, y_n) e^{i(p_1 y_1 + \dots + p_n y_n)} \quad (5.62)$$

$$= M^\nu(-ip_1, \dots, -ip_n) e^{i(p_1 + \dots + p_n)x}. \quad (5.63)$$

If we integrate over  $x$  and use (5.61) we find:

$$\int d^4x \widehat{a^\nu}(x, p_1, \dots, p_n) = (2\pi)^4 M^\nu(-ip_1, \dots, -ip_n) \delta(p_1 + \dots + p_n) = 0, \quad (5.64)$$

$$\Rightarrow M^\nu(-ip_1, \dots, -ip_n) \Big|_{p_1 + \dots + p_n = 0} = 0. \quad (5.65)$$

---

<sup>6</sup>One has to give some meaning to the formal integral in (5.56). We refer to the method of Requardt [Req76]. This shows that a charge like  $P^\mu$  can be defined for massive theories in general and for certain massless theories, if the infrared behaviour is not “too bad”. We explicitly show the existence in section 4 for the massless  $\varphi^4$ -model. But the same conclusion also holds if the mass dimension of  $\Theta_{0\text{can}}^{\mu\nu}$  is not less than four. This is the case here, if the fields in (5.24) contracted with  $K$  are bosonic and the ones contracted with  $G$  are fermionic, as usual.

<sup>7</sup>Note that the RHS of (5.54) is of compact support in the difference variables  $y_i - x$ .



We set  $q = \sum_{i=1}^n p_i$ , and write  $\widetilde{M}^\nu(q, p_1, \dots, p_{n-1}) = M^\nu(-ip_1, \dots, -ip_n)$ . Performing a Taylor expansion at the origin we get:

$$\widetilde{M}^\nu(q, p_1, \dots, p_{n-1}) = \sum_{k=1}^{\text{degree } \widetilde{M}^\nu} \sum_{|\alpha|+|\beta|=k} \frac{q^\alpha p^\beta}{\alpha! \beta!} \partial_\alpha^q \partial_\beta^p \widetilde{M}^\nu(0, \dots, 0), \quad (5.66)$$

$p = (p_1, \dots, p_{n-1})$ . Because of (5.65) there are only terms with  $|\alpha| \geq 1$  in the second sum. If we Fourier transform back into coordinate space this implies (5.55).

The normalization term  $M_1^{\mu\nu}$  cannot be added to the first two terms of  $T(\Theta_{0\text{can}}^{\mu\nu})$  only. The next two terms (conf. (5.24)) are multiples of the traces of the first ones. This symmetry has to be preserved since we demand linearity of the  $T$ -products:

$$\eta_{\mu\nu} T(\partial^\mu \varphi_j \partial^\nu \varphi_l, N) = T(\partial^\mu \varphi_j \partial_\mu \varphi_l, N) \quad (5.67)$$

$$\Rightarrow \eta_{\mu\nu} (\partial^\mu \varphi_j \partial^\nu \varphi_l)_{g\mathcal{L}} = (\partial^\mu \varphi_j \partial_\mu \varphi_l)_{g\mathcal{L}}. \quad (5.68)$$

Therefore we add the normalization terms according to

$$K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial^\nu \varphi_k, N) \rightarrow K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial^\nu \varphi_k, N) + \left( M_1^{\mu\nu} - \frac{1}{6} \eta^{\mu\nu} M_{1\rho}^\rho \right) (\partial) \delta, \quad (5.69)$$

$$\Rightarrow K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial_\mu \varphi_k, N) \rightarrow K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial_\mu \varphi_k, N) + \frac{1}{3} M_{1\rho}^\rho (\partial) \delta. \quad (5.70)$$

If the normalization has to be performed on the other terms we put:

$$G_{lk}^\mu T(\partial^\nu \varphi_l \varphi_k, N) \rightarrow G_{lk}^\mu T(\partial^\nu \varphi_l \varphi_k, N) + 2 \left( M_1^{\mu\nu} - \frac{1}{3} \eta^{\mu\nu} M_{1\rho}^\rho \right) (\partial) \delta, \quad (5.71)$$

$$\Rightarrow G_{lk}^\mu T(\partial_\mu \varphi_l \varphi_k, N) \rightarrow G_{lk}^\mu T(\partial_\mu \varphi_l \varphi_k, N) - \frac{2}{3} M_{1\rho}^\rho (\partial) \delta. \quad (5.72)$$

These normalizations remove the anomaly.

#### 4. The interacting momentum operator

Now we investigate the interacting charge generated by the conserved energy momentum tensor. It defines the interacting momentum operator since its commutator implements the infinitesimal action on  $\mathfrak{A}_{g\mathcal{L}}(\mathcal{O})$  according to:

$$[\Theta_{\text{can } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] = i \partial^\nu W_{g\mathcal{L}}(x) \quad (5.73)$$

for all  $W \in \mathfrak{B}$  containing no derivated fields. The test function  $f \in \mathcal{D}(\mathbb{M})$  is supposed to be  $f(y) = h(y^0)$  for all  $y = (y^0, \mathbf{y}) \in \mathbb{M}$  in a neighbourhood of  $x + (\overline{V}_+ \cup \overline{V}_-)$  and  $\int dy^0 h(y^0) = 1$ .

**4.1. Proof of (5.73).** We follow the idea of [DF99]. We use the abbreviation  $\mathrm{d}y_N g(y_N) \doteq \prod_{i \in N} \mathrm{d}y_i g(y_i)$ . With the definition of the commutator (4.20) the support properties of the  $R$ -products and the choice of  $f$  from above we have:

$$\begin{aligned}
[\Theta_{0\text{can } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] &= \int \mathrm{d}y h(y^0) [\Theta_{0\text{can } g\mathcal{L}}^{0\nu}(y), W_{g\mathcal{L}}(x)] = \\
&= - \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N \mathrm{d}y g(y_N) \times \\
&\quad \times \left[ (h(y^0) - h(y^0 - a) + h(y^0 - a)) R(N, \Theta_{0\text{can}}^{0\nu}; W)(y_N, y; x) + \right. \\
&\quad \left. - (h(y^0) - h(y^0 - b) + h(y^0 - b)) R(N, W; \Theta_{0\text{can}}^{0\nu})(y_N, x; y) \right], \tag{5.74}
\end{aligned}$$

where  $W_i = \mathcal{L}, i = 1, \dots, n$ . Since  $h$  has compact support we can choose  $a > 0$  and  $b < 0$  big enough that the contributions of  $h(y^0 - a)$  and  $h(y^0 - b)$  vanish due to the support properties of the  $R$ -products, see figure 1. If we define the

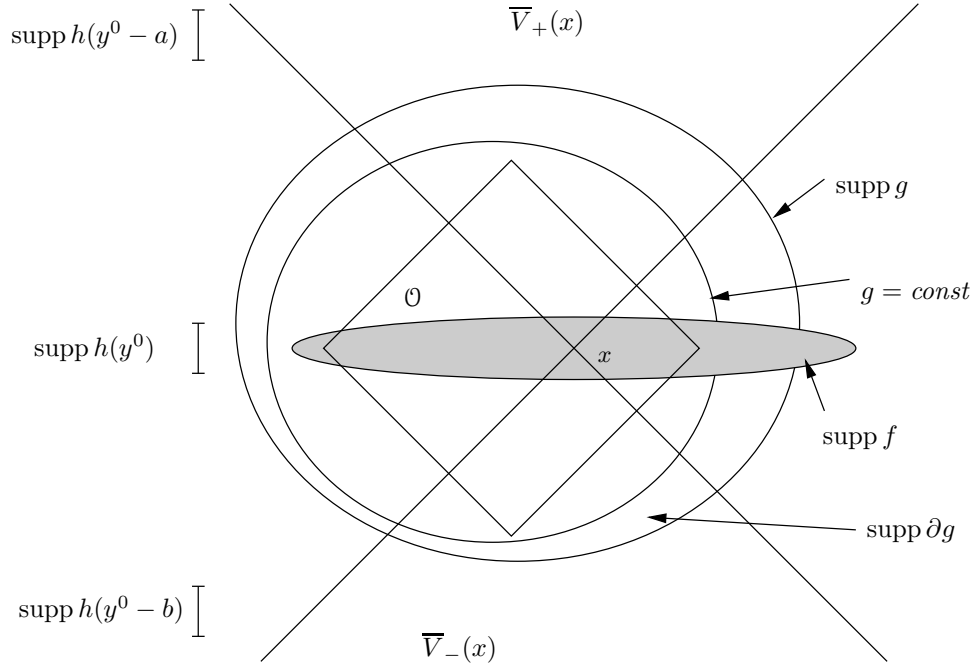


FIGURE 1. Supports of  $f, h, g, \partial g$  and  $R$ -products.

following two functions

$$k(y) \doteq k(y^0) = \int_{-\infty}^{y^0} \mathrm{d}z (h(z) - h(z - a)), \tag{5.75}$$

$$\tilde{k}(y) \doteq \tilde{k}(y^0) = \int_{y^0}^{\infty} \mathrm{d}z (h(z) - h(z - b)), \tag{5.76}$$

the commutator becomes

$$\begin{aligned}
& [\Theta_{0\text{can}}^{0\nu} g_{\mathcal{L}}(f), W_{g\mathcal{L}}(x)] = \\
& = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_N dy g(y_N) \times \\
& \quad \times \left[ k(y) \partial_{\mu}^y R(N, \Theta_{0\text{can}}^{\mu\nu}; W)(y_N, y; x) + \tilde{k}(y) \partial_{\mu}^y R(N, W; \Theta_{0\text{can}}^{\mu\nu})(y_N, x; y) \right] \\
& = i(k(x) + \tilde{k}(x)) \partial^{\nu} W_{g\mathcal{L}}(x) + \\
& \quad + i \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dy_N g(y_N) \times \\
& \quad \times \sum_{j=1}^n \left[ k(y_j) \partial_j^{\nu} R(N; W)(y_N; x) + \tilde{k}(y_j) \partial_j^{\nu} R(N \setminus j, W; j)(y_{N \setminus j}, x; y_j) \right] \\
& = i \partial^{\nu} W_{g\mathcal{L}}(x) + \\
& \quad - i \eta^{0\nu} \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dy_N g(y_N) \times \\
& \quad \times \sum_{j=1}^n \left[ (h(y_j^0) - h(y_j^0 - a)) R(N; W)(y_N; x) + \right. \\
& \quad \left. - (h(y_j^0) - h(y_j^0 - b)) R(N \setminus j, W; j)(y_{N \setminus j}, x; y_j) \right] + \\
& \quad - i \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dy_N \sum_{j=1}^n g(y_1) \dots \partial^{\nu} g(y_j) \dots g(y_n) \times \\
& \quad \times \left[ k(y_j) R(N; W)(y_N; x) + \tilde{k}(y_j) R(N \setminus j, W; j)(y_{N \setminus j}, x; y_j) \right], \tag{5.77}
\end{aligned}$$

where we have inserted the Ward identities (5.32), (5.36) and used the fact that

$$k(x) + \tilde{k}(x) = \int dz h(z) - \int_{-\infty}^{x^0} dz h(z - a) - \int_{x^0}^{\infty} dz h(z - b) = 1. \tag{5.78}$$

The support of  $f$  in the time direction and therefore of  $h$  can be made sufficiently small, see figure 1. Then the term  $\partial^{\nu} g(y_j) k(y_j) R(N; W)(y_N; x)$  vanishes due to the support of properties. The same is true for the second term in the last integrand of (5.77). In the first integrand the  $h(y_j^0 - a)$  and  $h(y_j^0 - b)$  can be

omitted, again. We obtain

$$\begin{aligned}
& [\Theta_{0\text{can } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] = \\
& = i\partial^\nu W_{g\mathcal{L}}(x) + \\
& \quad - i\eta^{0\nu} \sum_{n=1}^{\infty} \frac{i^n}{n!} \int dy_N g(y_N) \times \\
& \quad \times \sum_{j=1}^n h(y_j^0) \left[ R(N; W)(y_N; x) - R(N \setminus j, W; j)(y_{N \setminus j}, x; y_j) \right] \\
& = i\partial^\nu W_{g\mathcal{L}}(x) + \\
& \quad + \eta^{0\nu} \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} \int dy_N \sum_{j=1}^n g(y_1) \dots g(y_j) h(y_j^0) \dots g(y_n) \times \\
& \quad \times \frac{1}{n} \left[ R(N \setminus j, j; W)(y_{N \setminus j}, y_j; x) - R(N \setminus j, W; j)(y_{N \setminus j}, x; y_j) \right] \\
& = i\partial^\nu W_{g\mathcal{L}}(x) - \eta^{0\nu} [\mathcal{L}_{g\mathcal{L}}(gf), W_{g\mathcal{L}}(x)],
\end{aligned} \tag{5.79}$$

since all terms in the sum of the last integrand are equal because of the symmetry. Hence we find:

$$\begin{aligned}
[\Theta_{0\text{can } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] &= [\Theta_{0\text{can } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] + \eta^{0\nu} [\mathcal{L}_{g\mathcal{L}}(gf), W_{g\mathcal{L}}(x)] \\
&= i\partial^\nu W_{g\mathcal{L}}(x).
\end{aligned} \tag{5.80}$$

We show that one can define a momentum operator according to

$$P_{g\mathcal{L}}^\nu \doteq \int d^3\mathbf{x} \Theta_{0\text{can } g\mathcal{L}}^{0\nu}(x^0, \mathbf{x}). \tag{5.81}$$

Due to the definition of interacting fields (4.11) we start investigating the free contribution. Following the method of Requardt [Req76] we consider the localized momentum operator:

$$\lambda P^\nu \doteq \int d^4x k_\lambda(x^0) h_\lambda(\mathbf{x}) \Theta_{0\text{can}}^{0\nu}(x^0, \mathbf{x}), \tag{5.82}$$

with test functions  $h \in \mathcal{D}(\mathbb{R}^3)$ ,  $h(0) \equiv 1$  and  $k \in \mathcal{D}(\mathbb{R})$  with  $\int dx^0 k(x^0) = 1$ . We set  $h_\lambda(\mathbf{x}) \doteq h(\lambda\mathbf{x})$  and  $k_\lambda(x^0) \doteq \lambda k(\lambda x^0)$ . Calculation of the correlation function of two EMTs results in:

$$\begin{aligned}
& \left\langle \Omega, \Theta_{0\text{can}}^{0\mu}(x) \Theta_{0\text{can}}^{0\nu}(y) \Omega \right\rangle = \\
& = \partial^0 \partial^0 D_+(x-y) \partial^\mu \partial^\nu D_+(x-y) + \partial^0 \partial^\mu D_+(x-y) \partial^0 \partial^\nu D_+(x-y) + \\
& \quad - 2\eta^{0(\mu} \partial^{\nu)} \partial^\rho D_+(x-y) \partial_\rho \partial^0 D_+(x-y) + \\
& \quad + \frac{1}{2} \eta^{0\mu} \eta^{0\nu} \partial_\rho \partial_\sigma D_+(x-y) \partial^\rho \partial^\sigma D_+(x-y) \\
& = \lambda^8 \left\langle \Omega, \Theta_{0\text{can}}^{0\mu}(\lambda x) \Theta_{0\text{can}}^{0\nu}(\lambda y) \Omega \right\rangle,
\end{aligned} \tag{5.83}$$

since the massless two-point function  $D_+$  is homogenous of degree  $-2$ . We have

$$\langle \Omega, {}_\lambda P^\mu {}_\lambda P^\nu \Omega \rangle = \lambda^2 \int d^4x d^4y k(x^0)k(y^0)h(\mathbf{x})h(\mathbf{y}) \left\langle \Omega, \Theta_{0\text{can}}^{0\mu}(x)\Theta_{0\text{can}}^{0\nu}(y)\Omega \right\rangle, \quad (5.84)$$

and this implies  $\lim_{\lambda \rightarrow 0} \| {}_\lambda P^\nu \Omega \| = 0$ . For an arbitrary Wick polynomial  $W$  we additionally have

$$\lim_{\lambda \rightarrow 0} [ {}_\lambda P^\nu, W(x) ] = \partial^\nu W(x). \quad (5.85)$$

The domain  $\mathcal{D}$  is the linear hull of all  $\Phi = W_1(f_1) \dots W_r(f_r)\Omega$ , with  $W_i$  Wick monomials and  $f_i$  test functions. With the derivation property of the commutator, this defines the momentum  $P^\nu = \lim_{\lambda \rightarrow 0} {}_\lambda P^\nu$  in a strong limit on  $\mathcal{D}$ :  $P\Phi\Omega = [P, \Phi]\Omega$ .

In the interacting contribution of (5.81) the space integral is restricted to the hypersurface of constant  $x^0$  intersecting  $\bar{V}_+(\text{supp } g)$  and hence compact because of the support properties of the  $R$ -products. The interacting canonical tensor further contains the interaction term (cf. (5.27)) which is localized. Hence the integral in (5.81) exists.

### 5. The interacting improved tensor in massless $\varphi^4$ -theory

This section treats the possibilities for defining an improved EMT. It results in the unavoidable appearance of the well known trace anomaly. The definition of a suitable improvement tensor requires the validity of a further differential equation involving interacting fields. This equation is proved by a corresponding Ward identity in the next section.

We consider the free massless scalar field  $\varphi$ . It satisfies the wave equation and the commutation relation:

$$\square\varphi = 0, \quad [\varphi(x), \varphi(y)] = iD(x-y), \quad (5.86)$$

for  $\varphi(x) = T(\varphi)(x)$ .  $D$  is the Pauli-Jordan distribution. The corresponding Feynman propagator is denoted by  $D^F$ . The free field allows to define a conserved and traceless improved EMT by the expression from classical field theory in form of Wick products:

$$:\Theta_{0\text{imp}}^{\mu\nu}: = :\Theta_{0\text{can}}^{\mu\nu}: - \frac{1}{3} :I^{\mu\nu}:, \quad (5.87)$$

$$:I^{\mu\nu}: = :\partial^\mu \varphi \partial^\nu \varphi: + :\varphi \partial^\mu \partial^\nu \varphi: - \eta^{\mu\nu} : \partial_\rho \varphi \partial^\rho \varphi: \quad (5.88)$$

$$= \partial^\mu : \varphi \partial^\nu \varphi: - \eta^{\mu\nu} \partial_\rho : \varphi \partial^\rho \varphi: \quad (5.89)$$

$$= \frac{1}{2} (\partial^\mu \partial^\nu - \eta^{\mu\nu} \square) : \varphi^2: . \quad (5.90)$$

It is well known that the dilatations can be implemented as a unitary symmetry on the invariant domain  $\mathcal{D}$  of Fock space  $U : \mathcal{D} \mapsto \mathcal{D}, U_\lambda \varphi(x) U_\lambda^{-1} = \lambda \varphi(\lambda x)$ . The infinitesimal transformation is given by the commutator of the dilatation charge  $Q_D^R = \int dx^0 \alpha(x^0) f_R(\mathbf{x}) D^0(x)$  for sufficiently large  $R$ , where  $\int dx^0 \alpha(x^0) = 1$  and  $f_R$  is a smooth version of the characteristic function of the ball of radius  $R$ . For large  $R$  the commutator becomes independent of  $\alpha$  and

$f_R$  [MR71]. The dilatation current is  $D_0^\mu(x) = x_\nu : \Theta_{0\text{imp}}^{\mu\nu}(x) :$ . Since the symmetry is conserved we have  $\lim_{R \rightarrow \infty} \omega_0([Q_D^R, W]) = 0$ , for any observable  $W = \int dx_1 \dots dx_n : W_1(x_1) : \dots : W_n(x_n) : f(x_1) \dots f(x_n)$  with  $W_i \in \mathfrak{B}, f_i \in \mathcal{D}(\mathbb{M})$ .<sup>8</sup>

Switching on the interaction, the field equation becomes

$$\square \varphi_{g\mathcal{L}} = -g \left( \frac{\partial \mathcal{L}}{\partial \varphi} \right)_{g\mathcal{L}}. \quad (5.91)$$

Since the model is just a special case of the general situation discussed in the last sections this leads to a locally conserved canonical EMT (5.24),(5.27),(5.28):

$$\Theta_{\text{can } g\mathcal{L}}^{\mu\nu} = (\partial^\mu \varphi \partial^\nu \varphi)_{g\mathcal{L}} - \frac{1}{2} \eta^{\mu\nu} (\partial_\rho \varphi \partial^\rho \varphi)_{g\mathcal{L}} + g \eta^{\mu\nu} \mathcal{L}_{g\mathcal{L}}. \quad (5.92)$$

In order to define an interacting improvement tensor based on (5.89) we require the following identity for some  $c \in \mathbb{R}, c \neq \frac{1}{4}$ :

$$\begin{aligned} \partial^\mu (\varphi \partial^\nu \varphi)_{g\mathcal{L}} &= \\ &= (\partial^\mu \varphi \partial^\nu \varphi)_{g\mathcal{L}} + (\varphi \partial^\mu \partial^\nu \varphi)_{g\mathcal{L}} - c \eta^{\mu\nu} (\varphi \square \varphi)_{g\mathcal{L}} - c \eta^{\mu\nu} g \left( \varphi \frac{\partial \mathcal{L}}{\partial \varphi} \right)_{g\mathcal{L}}. \end{aligned} \quad (5.93)$$

The equation is obviously symmetric in  $\mu, \nu$ . We show this identity to be satisfied by a corresponding Ward identity in the next section. The exclusion of the case  $c = \frac{1}{4}$  is due to the fact that (5.93) has to be satisfied with **WI 1** simultaneously. Since the latter one fixes the normalization of  $(\partial^\mu \varphi \partial^\nu \varphi)_{g\mathcal{L}}$  the new interacting field  $(\varphi \partial^\mu \partial^\nu \varphi)_{g\mathcal{L}}$  must not appear in a traceless combination.

Now the improvement tensor is defined by

$$I_{g\mathcal{L}}^{\mu\nu} \doteq \partial^\mu (\varphi \partial^\nu \varphi)_{g\mathcal{L}} - \eta^{\mu\nu} \partial_\rho (\varphi \partial^\rho \varphi)_{g\mathcal{L}}. \quad (5.94)$$

It is conserved due to the  $\mu, \nu$ -symmetry of (5.93).

$$\partial_\mu I_{g\mathcal{L}}^{\mu\nu} = 0. \quad (5.95)$$

To discuss the consequences of the improvement we introduce the dimension operator  $d$  on monomials  $W \in \mathfrak{B}$  according to:

$$dW \doteq \sum_r (r+1) \varphi_{,\mu_1 \dots \mu_r} \frac{\partial W}{\partial \varphi_{,\mu_1 \dots \mu_r}}. \quad (5.96)$$

The 1 in parenthesis refers to the dimension of the scalar field  $\varphi$ . Obviously,  $d$  has integer eigenvalues. In case of a pure  $\mathcal{L} \propto \varphi^4$ -interaction we have:

$$d\mathcal{L} = \varphi \frac{\partial \mathcal{L}}{\partial \varphi} = 4\mathcal{L}. \quad (5.97)$$

Now we define the interacting improved EMT according to (5.15) by:

$$\Theta_{\text{imp } g\mathcal{L}}^{\mu\nu} \doteq \Theta_{\text{can } g\mathcal{L}}^{\mu\nu} - \frac{1}{3} I_{g\mathcal{L}}^{\mu\nu}. \quad (5.98)$$

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<sup>8</sup>Instead of writing  $\lim_{R \rightarrow \infty} [Q_D^R, W]$  or using the better convergent definition by Requardt [Req76] we use the expression  $[D^0(f), W]$  with a suitable test function  $f$ .

But the trace of that tensor is not zero. We find:

$$\eta_{\mu\nu}\Theta_{\text{imp } g\mathcal{L}}^{\mu\nu} = -(\partial_\rho\varphi\partial^\rho\varphi)_{g\mathcal{L}} + 4g\mathcal{L}_{g\mathcal{L}} + \partial_\rho(\varphi\partial^\rho\varphi)_{g\mathcal{L}} \quad (5.99)$$

$$= (1 - 4c) \left( (\varphi\Box\varphi)_{g\mathcal{L}} + g \left( \varphi \frac{\partial\mathcal{L}}{\partial\varphi} \right)_{g\mathcal{L}} \right). \quad (5.100)$$

This is the well known trace anomaly of the EMT. We see that it is undetermined up to a multiplicative real parameter. The anomaly is zero in case that one of the factors vanishes. But this contradicts the non existence of a scale invariant renormalization.<sup>9</sup>

The improved EMT defines the same momentum operator:

$$[\Theta_{\text{imp } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] = [\Theta_{\text{can } g\mathcal{L}}^{0\nu}(f), W_{g\mathcal{L}}(x)] = \partial^\nu W_{g\mathcal{L}}(x), \quad (5.111)$$

---

<sup>9</sup>To show the contradiction we have to treat the two cases

$$(i) \quad c = \frac{1}{4} \text{ and} \quad (5.101)$$

$$(ii) \quad (\varphi\Box\varphi)_{g\mathcal{L}} = -g \left( \varphi \frac{\partial\mathcal{L}}{\partial\varphi} \right)_{g\mathcal{L}}. \quad (5.102)$$

Assume (i) is true. With

$$I_0^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + \varphi\partial^\mu\partial^\nu\varphi - \eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \frac{1}{4}\eta^{\mu\nu}\varphi\Box\varphi, \quad (5.103)$$

$$I_{g\mathcal{L}}^{\mu\nu} = I_0^{\mu\nu} + \frac{3}{4}\eta^{\mu\nu}g(d\mathcal{L})_{g\mathcal{L}}, \quad (5.104)$$

we could define the locally conserved dilatation current by

$$D_{g\mathcal{L}}^\mu \doteq x_\nu\Theta_{\text{imp } g\mathcal{L}}^{\mu\nu} = x_\nu \left( \Theta_{\text{can } g\mathcal{L}}^{\mu\nu} - \frac{1}{3}I_{g\mathcal{L}}^{\mu\nu} \right) \quad (5.105)$$

$$= x_\nu \left( \Theta_{0\text{can } g\mathcal{L}}^{\mu\nu} - \frac{1}{3}I_{0g\mathcal{L}}^{\mu\nu} \right) \quad (5.106)$$

$$= \left( x_\nu \left( \Theta_{0\text{can}}^{\mu\nu} - \frac{1}{3}I_0^{\mu\nu} \right) \right)_{g\mathcal{L}}. \quad (5.107)$$

The conservation is equivalent to the Ward identity

$$\partial_\mu^\alpha T(D^\mu, N)(x, y_N) = i \sum_{k=0}^n \delta(x - y_k)(d_k + y_k \cdot \partial_k)T(N)(y_N), \quad (5.108)$$

where  $d_k$  denotes the  $k$ 'th dimension (see next footnote). Passing to the integrated Ward identity by integration with a function  $g$  chosen like in step 3 of section 3 this leads to

$$[D^0(f), T(N)(y_N)] = \sum_{k=1}^n (d_k + y_k \cdot \partial^k)T(N)(y_N), \quad (5.109)$$

where  $f$  is a test function like in section 4. Since the dilatations are the infinitesimal symmetry transformations of the unitarily implementable scale transformations on the free field algebra, the rhs vanishes in the vacuum state  $\omega_0$ . But this implies a scale invariant renormalization of  $\omega_0(T(N))$  which is not possible. Therefore  $c \neq \frac{1}{4}$ .

If we assume (ii) to be true, the last two terms of (5.93) vanish leading to the conserved improvement tensor  $I_{g\mathcal{L}}^{\mu\nu}$  with

$$I^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + \varphi\partial^\mu\partial^\nu\varphi - \eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \eta^{\mu\nu}\varphi\Box\varphi. \quad (5.110)$$

The dilatations according to (5.105) are conserved and both Ward identities (5.108), (5.109) hold with  $D^\mu$  replaced by  $D_0^\mu = x_\nu(\Theta_{0\text{can}}^{\mu\nu} - \frac{1}{3}I_0^{\mu\nu})$ . Because of the same argument this is a contradiction.

because the  $I_{g\mathcal{L}}^{0\nu}$ -component is either a derivative (or divergence) w.r.t. to the space components and  $\partial_j f(y) = 0, y \in \overline{V}_+(x) \cup \overline{V}_-(x)$ :

$$I_{g\mathcal{L}}^{00} = \partial_j (\varphi \partial^j \varphi)_{g\mathcal{L}}, \quad (5.112)$$

$$I_{g\mathcal{L}}^{0j} = \partial^j (\varphi \partial^0 \varphi)_{g\mathcal{L}}. \quad (5.113)$$

The trace of the improved EMT expresses the breaking of scale invariance. If the dilatations are defined by (cp. (5.18))

$$D_{g\mathcal{L}}^\mu = x_\nu \Theta_{\text{imp } g\mathcal{L}}^{\mu\nu}, \quad (5.114)$$

we obtain

$$\partial_\mu D_{g\mathcal{L}}^\mu = \Theta_{\text{imp } g\mathcal{L}}^\mu{}_\mu + x^\mu \partial_\mu g\mathcal{L}_{g\mathcal{L}}. \quad (5.115)$$

If we define the dilatations alternatively by (cp. (5.20))

$$\tilde{D}_{g\mathcal{L}}^\mu = x_\nu \Theta_{\text{can } g\mathcal{L}}^{\mu\nu} + (\varphi \partial^\mu \varphi)_{g\mathcal{L}}, \quad (5.116)$$

we find the same breaking (5.115). The (not time independent) charge remains unchanged due to:

$$\tilde{D}_{g\mathcal{L}}^0 - D_{g\mathcal{L}}^0 = \frac{2}{3} \partial_j \left( x^{[j} (\varphi \partial^{0]} \varphi)_{g\mathcal{L}} \right). \quad (5.117)$$

The next section gives a proof of (5.93).

## 6. Proof of the Ward identity

We prove equation (5.93) in analogy to the conservation of the canonical EMT by the validity of the following Ward identity:<sup>10</sup>

$$\begin{aligned} \partial_x^\mu T(\varphi \partial^\nu \varphi, N)(x, y_N) &= T(\partial^\mu \varphi \partial^\nu \varphi, N)(x, y_N) + T(\varphi \partial^\mu \partial^\nu \varphi, N)(x, y_N) + \\ &\quad - c\eta^{\mu\nu} T(\varphi \square \varphi, N)(x, y_N) + i c \eta^{\mu\nu} \sum_{k=1}^n \delta(y_k - x) d_k T(N)(y_N). \quad (\mathbf{WI} \ 2) \end{aligned}$$

The proof follows the procedure in section 3. For  $N = \emptyset$  **WI 2** is obviously fulfilled. Then we make a double induction over  $n = |N|$  and the degree of the Wick monomials. Under the assumption that **WI 2** is fulfilled in lower orders we denote the possible anomaly by

$$\begin{aligned} a^{\mu\nu}(N)(x, y_N) &\doteq \partial_x^\mu T(\varphi \partial^\nu \varphi, N)(x, y_N) - T(\partial^\mu \varphi \partial^\nu \varphi, N)(x, y_N) + \\ &\quad - T(\varphi \partial^\mu \partial^\nu \varphi, N)(x, y_N) + c\eta^{\mu\nu} T(\varphi \square \varphi, N)(x, y_N) + \\ &\quad - i\eta^{\mu\nu} c \sum_{k=1}^n \delta(y_k - x) d_k T(N)(y_N). \quad (5.118) \end{aligned}$$

We show that a normalization of the  $T(\varphi \partial^\mu \partial^\nu \varphi, N)$  exists such that the anomaly vanishes if we require the following normalization condition for the twice derivated basic field:

$$T(\partial_\mu \partial_\nu \varphi, N)(x, y_N) \doteq \partial_\mu^x \partial_\nu^x T(\varphi, N)(x, y_N). \quad (5.119)$$

---

<sup>10</sup>  $d_k$  is the dimension of the  $k$ 'th monomial in the  $T$ -product:  $d_k T(N) = T(W_1, \dots, dW_k, \dots, W_n)$  and  $d$  given by (5.96).



By comparison to (3.60), the integrated form of **N4**, this is achieved by fixing the two point function:  $\omega_0(T(\partial^\mu \partial^\nu \varphi, \varphi)(x, y)) = i\partial^\mu \partial^\nu D^F(x - y)$ . For the corresponding interacting fields we have  $(\partial^\mu \partial^\nu \varphi)_{g\mathcal{L}} = \partial^\mu \partial^\nu \varphi_{g\mathcal{L}}$ . In order to condense the notation we introduce the following abbreviation:

$$N^{(e_k)} \doteq \left\{ W_1, \dots, \frac{\partial W_k}{\partial \varphi}, \dots, W_n \right\}. \quad (5.120)$$

**Step 1.** We commute the anomaly with the free field  $\varphi(z)$ :

$$\begin{aligned} [a^{\mu\nu}(N)(x, y_N), \varphi(z)] &= \\ &= \partial_x^\mu [T(\varphi \partial^\nu \varphi, N)(x, y_N), \varphi(z)] - [T(\partial^\mu \varphi \partial^\nu \varphi, N)(x, y_N), \varphi(z)] + \\ &\quad - [T(\varphi \partial^\mu \partial^\nu \varphi, N)(x, y_N), \varphi(z)] + c\eta^{\mu\nu} [T(\varphi \square \varphi, N)(x, y_N), \varphi(z)] + \\ &\quad - i c \eta^{\mu\nu} \sum_{l=1}^n \delta(y_l - x) d_k [T(N)(y_N), \varphi(z)] \\ &= \partial_x^\mu \partial_x^\nu T(\varphi, N)(x, y_N) i D(x - z) + \partial_x^\nu T(\varphi, N)(x, y_N) \partial^\mu D(x - z) + \\ &\quad + \partial_x^\mu T(\varphi, N)(x, y_N) i \partial^\nu D(x - z) + T(\varphi, N)(x, y_N) i \partial^\mu \partial^\nu D(x - z) + \\ &\quad - \partial_x^\nu T(\varphi, N)(x, y_N) i \partial^\mu D(x - z) - \partial_x^\mu T(\varphi, N)(x, y_N) i \partial^\nu D(x - z) + \\ &\quad - T(\partial^\mu \partial^\nu \varphi, N)(x, y_N) i D(x - z) - T(\varphi, N)(x, y_N) i \partial^\mu \partial^\nu D(x - z) + \\ &\quad + c\eta^{\mu\nu} T(\square \varphi, N)(x, y_N) i D(x - z) + \\ &\quad - i c \eta^{\mu\nu} \sum_{k=1}^n \delta(y_k - x) T(N^{(e_k)})(y_N) i D(y_k - z) + \\ &\quad + \sum_{k=1}^n a^{\mu\nu}(N^{(e_k)})(x, y_N) i D(y_k - z) \\ &= 0, \end{aligned} \quad (5.121)$$

if we apply the induction assumption ( $a^{\mu\nu}(N^{(e_k)}) = 0$ ) to the last line and the normalization (5.119) and normalization condition **N4** to the previous lines. Therefore, the anomaly has to be a vacuum expectation value:

$$a^{\mu\nu}(N)(x, y_N) = \omega_0(a^{\mu\nu}(N)(x, y_N)). \quad (5.122)$$

**Step 2.** We show that the anomaly vanishes if one Wick monomial is a basic generator  $\varphi$ . Because of Step 1 we only consider vacuum expectation values and

use (3.60):

$$\begin{aligned}
\omega_0(a^{\mu\nu}(N, \varphi)(x, y_N, z)) &= \\
&= \partial_x^\mu \omega_0(T(\varphi \partial^\nu \varphi, N, \varphi)(x, y_N, z)) + \\
&\quad - \omega_0(T(\partial^\mu \varphi \partial^\nu \varphi, N, \varphi)(x, y_N, z)) + \\
&\quad - \omega_0(T(\varphi \partial^\mu \partial^\nu \varphi, N, \varphi)(x, y_N, z)) + \\
&\quad + c\eta^{\mu\nu} \omega_0(T(\varphi \square \varphi, N, \varphi)(x, y_N, z)) + \\
&\quad - i c \eta^{\mu\nu} \sum_{l=1}^n \delta(y_l - x) \omega_0(d_l T(N, \varphi)(y_N, z)) + \\
&\quad - i c \eta^{\mu\nu} \delta(z - x) \omega_0(T(N, \varphi)(y_N, z)) \\
&= -i \partial^\mu D^F(z - x) \omega_0(T(\partial^\nu \varphi, N)(x, y_N)) + \\
&\quad + i D^F(z - x) \partial_x^\mu \omega_0(T(\partial^\nu \varphi, N)(x, y_N)) + \\
&\quad + i \partial^\mu \partial^\nu D^F(z - x) \omega_0(T(\varphi, N)(x, y_N)) + \\
&\quad - i \partial^\nu D^F(z - x) \partial_x^\mu \omega_0(T(\varphi, N)(x, y_N)) + \\
&\quad + i \partial^\mu D^F(z - x) \omega_0(T(\partial^\nu \varphi, N)(x, y_N)) + \\
&\quad + i \partial^\nu D^F(z - x) \omega_0(T(\partial^\mu \varphi, N)(x, y_N)) + \\
&\quad - i D^F(z - x) \omega_0(T(\partial^\mu \partial^\nu \varphi, N)(x, y_N)) + \\
&\quad - i \partial^\mu \partial^\nu D^F(z - x) \omega_0(T(\varphi, N)(x, y_N)) + \\
&\quad + i c \eta^{\mu\nu} D^F(z - x) \omega_0(T(\square \varphi, N)(x, y_N)) + \\
&\quad + i c \eta^{\mu\nu} \square D^F(z - x) \omega_0(T(\varphi, N)(x, y_N)) + \\
&\quad - i c \eta^{\mu\nu} \sum_{k=1}^n \delta(x - y_k) \omega_0(T(N^{(e_k)})(y_N)) i D^F(z - y_k) + \\
&\quad - i c \eta^{\mu\nu} \delta(z - x) \sum_{k=1}^n i D^F(z - y_k) \omega_0(T(N^{(e_k)})(y_N)) + \\
&\quad + \sum_{k=1}^n \omega_0(a^{\mu\nu}(N^{(e_k)})(x, y_N)) i D^F(z - y_k) \\
&= 0.
\end{aligned} \tag{5.123}$$

We have used  $\square D^F = \delta$ , the normalization (5.119), **N4** and the induction assumption.

**Step 3.** Because of the induction assumption the anomaly is a local term:

$$\omega_0(a^{\mu\nu}(N)(x, y)) = M^{\mu\nu}(\partial) \delta(x, y_N), \tag{5.124}$$

where  $\delta(x, y_N) = \delta(x - y_1) \dots \delta(x - y_n)$  and  $M^{\mu\nu}$  is a polynomial in  $\partial = (\partial_1, \dots, \partial_n)$  of degree  $\leq 4$ . Therefore we can absorb the anomaly by the following normalization:

$$\omega_0(T(\varphi \partial^\mu \partial^\nu \varphi, N)(x, y_N)) \rightarrow \omega_0(T(\varphi \partial^\mu \partial^\nu \varphi, N)(x, y_N)) + M^{\mu\nu}(\partial) \delta(x, y_N) +$$

$$- \frac{c}{4c-1} \eta^{\mu\nu} M^\rho_\rho(\partial) \delta(x, y_N), \quad (5.125)$$

$$\Rightarrow \omega_0(T(\varphi \square \varphi, N)(x, y_N)) \rightarrow \omega_0(T(\varphi \square \varphi, N)(x, y_N)) - \frac{1}{4c-1} M^\rho_\rho(\partial) \delta(x, y_N), \quad (5.126)$$

$$\Rightarrow \omega_0(a^{\mu\nu}(N)(x, y_N)) \rightarrow 0. \quad (5.127)$$

As long as the trace of the anomaly  $M^\rho_\rho$  is non vanishing the normalization can only be done for  $c \neq \frac{1}{4}$ .

## 7. The anomalous dimension

Although the dilatation current is not conserved in the interacting theory we calculate the commutator of the corresponding charge in order to obtain a term that measures the anomalous dimension. From **WI 2** we derive the following Ward identities for the  $R$ -products:

$$\partial_\mu^y R(N, \varphi \partial^\mu \varphi; W)(y_N, y; x) =$$

$$= R(N, \partial_\mu \varphi \partial^\mu \varphi; W)(y_N, y; x) + (1 - 4c) R(N, \varphi \square \varphi; W)(y_N, y; x) +$$

$$+ 4ic \sum_{j \in N} \delta(y_j - y) d_j R(N; W)(y_N; x) + 4ic \delta(x - y) R(N; dW)(y_N; x). \quad (5.128)$$

If we set  $M = \{V_1, \dots, V_m\}, V_i \in \mathfrak{B}$  we find:

$$\partial_\mu^x R(N; \varphi \partial^\mu \varphi, M)(y_N; x, x_M) =$$

$$= R(N, \partial_\mu \varphi \partial^\mu \varphi, M)(y_N; x, x_M) + (1 - 4c) R(N; \varphi \square \varphi, M)(y_N; x, x_M) +$$

$$+ 4ic \sum_{j \in N} \delta(y_j - x) d_j R(N \setminus j; j, M)(y_{N \setminus j}; y_j, x_M) +$$

$$+ 4ic \sum_{j \in M} \delta(x_j - x) d_j R(N; M)(y_N; x_M). \quad (5.129)$$

A calculation similar to the one given in section 4 shows that the following commutation relation holds:

$$\begin{aligned}
[D_{g\mathcal{L}}^0(f), W_{g\mathcal{L}}(x)] &= \\
&= i(dW)_{g\mathcal{L}}(x) + ix^\mu \partial_\mu W_{g\mathcal{L}}(x) + \\
&\quad + (1 - 4c) \left\{ \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_N dy g(y_N) \times \right. \\
&\quad \times \left[ \left( R(N, gd\mathcal{L}; W)(y_N, y; x) + R(N, \varphi \square \varphi; W)(y_N, y; x) \right) k(y) + \right. \\
&\quad + \left. \left( R(N, W; gd\mathcal{L})(y_N, x; y) + R(N, W; \varphi \square \varphi)(y_N, x; y) \right) \tilde{k}(y) \right] + \\
&\quad \left. - i(dW)_{g\mathcal{L}}(x) \right\}. \tag{5.130}
\end{aligned}$$

The terms in braces are the anomalous contributions. They are necessarily non vanishing (because of the normalization that excludes case (ii) in (5.102)). Moreover they are operator valued. We compute them for the case  $W = \varphi$ , where the normalization of the  $R$ -products is known due to **N4** [DF99].

$$\begin{aligned}
[D_{g\mathcal{L}}^0(f), \varphi_{g\mathcal{L}}(x)] &= \\
&= i\varphi_{g\mathcal{L}}(x) + ix^\mu \partial_\mu \varphi_{g\mathcal{L}}(x) + \\
&\quad + i(1 - 4c) \left\{ \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dy_N dy g(y_N) \times \right. \\
&\quad \times \left\{ \sum_{l=1}^n \left[ D_{\text{ret}}(x - y_l) \left( g(y) R \left( N \setminus l, d\mathcal{L}; \frac{\partial \mathcal{L}}{\partial \varphi} \right) (y_{N \setminus l}, y; y_l) + \right. \right. \\
&\quad \quad + R \left( N \setminus l, \varphi \square \varphi; \frac{\partial \mathcal{L}}{\partial \varphi} \right) (y_{N \setminus l}, y; y_l) \right) k(y) + \\
&\quad \quad + D_{\text{av}}(x - y_l) \left( g(y) R \left( N^{(e_l)}; d\mathcal{L} \right) (y_N; y) + \right. \\
&\quad \quad + R \left( N^{(e_l)}; \varphi \square \varphi \right) (y_N; y) \right) \tilde{k}(y) \right] + \\
&\quad \left. + \left( D_{\text{ret}}(x - y) k(y) + D_{\text{av}}(x - y) \tilde{k}(y) \right) R \left( N; \varphi \frac{\partial^2 \mathcal{L}}{\partial \varphi^2} \right) (y_N; y) k(y) \right\} \right\}. \tag{5.131}
\end{aligned}$$

On the other hand we can study the interacting Ward identities of the dilations. Since time ordered products of interacting fields are already determined by time ordered products of free fields with an arbitrary number of interactions

according to (4.21), we find with (5.128), (5.129):

$$\begin{aligned}
\partial_\mu^x T \left( D_{g\mathcal{L}}^\mu, \varphi, \dots, \varphi \right)_{g\mathcal{L}} (x, x_1, \dots, x_m) = \\
= T \left( \Theta_{\text{imp}}^\mu, \varphi, \dots, \varphi \right)_{g\mathcal{L}} (x, x_1, \dots, x_m) + \\
+ x^\mu \partial_\mu g(x) T(\mathcal{L}, \varphi, \dots, \varphi)_{g\mathcal{L}} (x, x_1, \dots, x_m) + \\
+ i \sum_{l=1}^m \delta(x_l - x) (4c + x_l^\mu \partial_\mu^l) T(\varphi, \dots, \varphi)_{g\mathcal{L}} (x_1, \dots, x_m).
\end{aligned} \tag{5.132}$$

The anomalous terms defined above are no anomalous dimensions in the form of a formal (local) power series in the coupling that multiply the interacting fields. This is no surprise since already for the free massive field the dilatations are no symmetry and the above method of commuting with the (non time invariant) dilatation charge does not produce a number in that case. Nevertheless also the massive scalar field is given the canonical dimension one. In [CJ71] the authors suggest to define the dimension by equal time commutators. For the free field this method is well defined and produces the right result. But it is not clear that this carries over to the interacting fields since they may become more singular objects in the time coordinate as their free counterparts.



## CHAPTER 6

### Operator product expansions

In [Wil69] Wilson suggested that a product of interacting field operators on separated points could be expanded into a sum of local operators if the separation goes to zero. Such an expansion is called *operator product expansion* (OPE).

Zimmermann has introduced his notion of perturbative normal products in [Zim71, Zim73a]. In his approach (interacting) operators are always defined via their Green's functions, namely vacuum expectation values of time ordered products with an arbitrary number of fields. The Green's functions are renormalized by BPHZ- or in the massless case by BPHZL subtraction.

In [Zim71, Zim73b] he gave a generalization of these local products to multi local ones. They admit a restriction of all coordinates to one yielding the local normal product. By relating a bilocal product to a time ordered one he derived an OPE verifying Wilson's hypothesis perturbatively. He found explicit formulas for the expansion coefficients in the form of Green's functions.

Since in the framework of Bogoliubov and Epstein-Glaser the operators are defined directly we try to mimic Zimmermann's procedure. We define a new time ordered product containing a bilocal expression which allows for setting its coordinates to the same value (in the sense of a restriction of a distribution). We are concerned with scalar fields only and our bilocal  $T$ -product has only the basic generators in the bilocal insertion.

The definition of a bilocal  $T$ -product gives rise to a corresponding bilocal interacting field. Following Zimmermann's notation we also call this object a normal product. The transition from the  $T$ -products to the interacting fields automatically generates an OPE for the time ordered product of two interacting fields. The coefficients depend on the coupling only locally. In  $\varphi^4$ -theory two coefficients appear. One consist of graphs that contribute to wave function and mass renormalization only. The other collects graphs contributing only to coupling constant renormalization.

With the normal product defined we investigate the first step towards the definition of a state on the local algebra. We find the corresponding two point function to be positive in an appropriate sense as a formal power series.

#### 1. Bilocal time ordered products

The word *bilocal time ordered product* means a usual time ordered product where only *one* entry is a bilocal expression. We derive an explicit formula that defines these products for two scalar fields. Let us mention that our expression is only explicit up to normalization terms which restore broken Lorentz covariance (cf. chapter 3, section 5). We state the problem first:

Consider the case of the interacting scalar fields  $\varphi_{g\mathcal{L}}$ . We aim at the definition of a *normal product*  $:\varphi, \varphi:_{g\mathcal{L}}(x_1, x_2)$  with the property

$$\lim_{\xi \rightarrow 0} :\varphi, \varphi:_{g\mathcal{L}}(x + \xi, x - \xi) = (\varphi^2)_{g\mathcal{L}}(x), \quad (6.1)$$

where  $x = \frac{x_1 + x_2}{2}$  denotes the central coordinate and  $\xi = \frac{x_1 - x_2}{2}$  the difference coordinate. Taking the definition of interacting fields into account (chapter 4), we notice that it suffices to define the corresponding  $T$ -products. This is illustrated in

**1.1. The easiest example.** We consider scalar  $\frac{\varphi^4}{4!}$ -theory. The task is to determine  $T(\mathcal{L}, : \varphi, \varphi :)(y, x_1, x_2)$ . We proceed in the following way: First we define  ${}^0T(\mathcal{L}, : \varphi, \varphi :)(y, x_1, x_2)$  which consists of the usual  $T$ -product with  $x_1, x_2$ -contractions omitted. Then we subtract a suitable term with support on  $y = x$  that allows for the restriction. Obviously we have

$${}^0T(\mathcal{L}, : \varphi, \varphi :)(y, x_1, x_2) = : \frac{\varphi(y)^4}{4!} \varphi(x_1) \varphi(x_2) : + \quad (6.2)$$

$$+ i\Delta_F(y - x_1) : \frac{\varphi(y)^3}{3!} \varphi(x_2) : \quad (6.3)$$

$$+ i\Delta_F(y - x_2) : \frac{\varphi(y)^3}{3!} \varphi(x_1) : + \quad (6.4)$$

$$+ i\Delta_F(y - x_1) i\Delta_F(y - x_2) : \frac{\varphi(y)^2}{2!} : . \quad (6.5)$$

The problem for the definition of the restriction of  ${}^0T$  emerges on the last line, where after setting  $x_1 = x_2 (= x)$  we obtain a  $\Delta_F(y - x)^2$ -term that is not well defined. Graphically this procedure produces a loop, see figure 1. But we

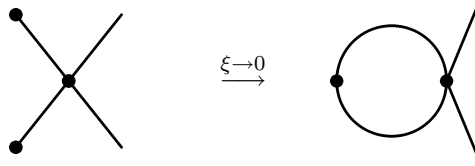


FIGURE 1. A loop is generated if  $\xi \rightarrow 0$ .

can find a term which subtraction allows for putting  $x_1 = x_2 = x$ , yielding  $T(\mathcal{L}, \varphi^2)(y, x)$  with its respective normalization. We claim that with

$$e^{(2)}(\xi) \doteq \int dz i\Delta^F(z - \xi) i\Delta^F(z + \xi) w(z) \quad (6.6)$$

the subtracted distribution

$$i\Delta^F(y - x_1) i\Delta^F(y - x_2) \Big|_R = i\Delta^F(y - x_1) i\Delta^F(y - x_2) - \delta(x - y) e^{(2)}(\xi) \quad (6.7)$$



allows for the coincidence  $x_1 = x_2 \Leftrightarrow \xi = 0$  after smearing with a test function. We calculate

$$\begin{aligned} \int dy i\Delta_F(y - x_1) i\Delta_F(y - x_2) \Big|_R f(y) &= \\ &= \int dy i\Delta_F(y - x_1) i\Delta_F(y - x_2) (f(y) - w(y - x)f(x)), \end{aligned} \quad (6.8)$$

and this implies

$$\lim_{\xi \rightarrow 0} \left\langle i\Delta^F(\cdot + \xi) i\Delta^F(\cdot - \xi) \Big|_R, f_x \right\rangle = \langle i^2(\Delta^F)^2, W_{(0;w)} f_x \rangle, \quad (6.9)$$

with  $f_x(y) = f(y + x)$ , which proves our claim. Note, that the coefficient in front of the  $\delta$  term is given by

$$e^{(2)}(\xi) = \int dz \omega_0 \left( {}^0T(\mathcal{L}^{(2)}, : \varphi, \varphi :)(z; \xi, -\xi) \right) w(z), \quad (6.10)$$

where the  $^{(2)}$  again denotes twice differentiation with respect to  $\varphi$ . Now we return to the complete  $T$ -product. Collecting everything into one expression we find

$$T(\mathcal{L}, : \varphi, \varphi :)(y, x_1, x_2) = {}^0T(\mathcal{L}, : \varphi, \varphi :)(y, x_1, x_2) - \delta(y - x) e^{(2)}(\xi) \frac{: \varphi(x)^2 :}{2}. \quad (6.11)$$

Because of the  $\delta$  distribution we have set the coordinate of the last Wick monomial to  $x$ .

**REMARK.** As well as the center coordinate  $x$  we could have chosen any other point on the straight line between  $x_1$  and  $x_2$  (or also in the causally completed region spanned by these two points) for subtraction. But our choice is inspired by Zimmermann's work, moreover yielding a symmetrical solution.

Now we generalize the idea of subtracting a local term that compensates the overall divergence. Thereby we make use of the method of Epstein-Glaser where all lower order divergencies are appropriately handled by an

**1.2. Inductive causal construction.** We begin with a brief overview of the construction. Motivated by our example above we define the bilocal  $T$ -products in the following way: Denoting the sub manifold

$$\text{Diag}_n^x \doteq \left\{ (y_1, \dots, y_n, x_1, x_2) \in \mathbb{M}^{n+2} \mid y_1 = \dots = y_n = \frac{x_1 + x_2}{2} \right\}, \quad (6.12)$$

we require the bilocal  $T$ -product of order  $n$  (where  $n$  is the number of the coordinates not including the two bilocal points) to be given by all bilocal  $T$ -products of lower and all local  $T$ -products of lower and same order on  $\mathbb{M}^{n+2} \setminus \text{Diag}_n^x$ . This provides for a  ${}^0T$ -product which yields the local  ${}^0T$ -product in the limit  $\xi \rightarrow 0$ . Then we subtract a term with support on  $\text{Diag}_n^x$  such that the limit  $\xi \rightarrow 0$  exists and yields the corresponding  $T$ -product. Hence in every order the difference between a local  $T$ -product (with  $x_1, x_2$ -contractions omitted) and a bilocal one only consists of these local terms.

In zero'th order we define:  $T(:\varphi, \varphi:)(x_1, x_2) \doteq \varphi(x_1)\varphi(x_2) :$ . Using our shorthand notation for the arguments ( $N$  can be any set of Wick monomials) we require the following causal factorization properties:

$$\begin{aligned}
T(N : \varphi, \varphi:)(y_N, x_1, x_2) = & \\
= & \begin{cases} T(I)(y_I)T(N \setminus I, : \varphi, \varphi:)(y_{N \setminus I}, x_1, x_2), & \text{if } I \gtrsim N \setminus I, x_1, x_2, I \neq \emptyset, \\ T(I, : \varphi, \varphi:)(y_I, x_1, x_2)T(N \setminus I)(y_{N \setminus I}), & \text{if } I, x_1, x_2, \gtrsim N \setminus I, I \neq N, \\ T(N, \varphi, \varphi)(y_N, x_1, x_2) + \\ + T(I)(y_I)[T(N \setminus I, : \varphi, \varphi:)(y_{N \setminus I}, x_1, x_2) + \\ - T(N \setminus I, \varphi, \varphi)(y_{N \setminus I}, x_1, x_2)], & \text{if } I, x_1 \gtrsim N \setminus I, x, x_2, I \neq \emptyset, \\ T(N, \varphi, \varphi)(y_N, x_1, x_2) + \\ + [T(I, : \varphi, \varphi:)(y_I, x_1, x_2) + \\ - T(I, \varphi, \varphi)(y_I, x_1, x_2)]T(N \setminus I)(y_{N \setminus I}), & \text{if } I, x_1, x, \gtrsim N \setminus I, x_2, I \neq N, \\ \text{the last two expressions with } x_1 \leftrightarrow x_2. \end{cases}
\end{aligned} \tag{6.13}$$

We convince ourselves that this is a reasonable causal factorization. If  $x_1, x_2$  are contracted to a point, then the first two equations obviously give the right causal decompositions. We investigate the last term on the third line, where  $I, x_1 \gtrsim N \setminus I, x, x_2$ . If  $\xi \rightarrow 0$  we find

$$\begin{aligned}
T(I)(y_I)T(N \setminus I, \varphi, \varphi)(y_{N \setminus I}, x_1, x_2) &= \\
&= T(I)(y_I)\varphi(x_1)T(N \setminus I, \varphi)(y_{N \setminus I}, x_2) \\
&\stackrel{\xi \rightarrow 0}{=} T(I, \varphi)(y_I, x_1)T(N \setminus I, \varphi)(y_{N \setminus I}, x_2) \tag{6.14} \\
&= T(N, \varphi, \varphi)(y_N, x_1, x_2),
\end{aligned}$$

since  $x_1$  becomes earlier than all  $y_I$ . This term cancels the first term of the third line in (6.13) leaving the first line of (6.13). A similar consideration leads to the same conclusion also for the last line of (6.13).

This shows that the bilocal  $T$ -product is completely determined by (6.13) up to the sub manifold  $\text{Diag}_n^x$ . In contrast to the definition of a local  $T$ -product where one has to perform an extension of the numerical distributions involved, the bilocal product can be defined by a suitable subtraction. This is due to the fact that all terms are well defined distributions in  $n+2$  variables. We state the solution:

$$\begin{aligned}
T(N, : \varphi, \varphi:)(y_N, x_1, x_2) &= \\
&= T(N, \varphi, \varphi)(y_N, x_1, x_2) - \omega_0(T(\varphi, \varphi)(x_1, x_2))T(N)(y_N) + \\
&\quad - \sum_{\substack{I \subset N \\ I \neq \emptyset}} \sum_{|\gamma| \leq \omega_I} \sum_{|\alpha| \leq \omega_I - |\gamma|} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \\
&\quad \times (e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi))T(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x),
\end{aligned} \tag{6.15}$$

where  $\gamma \in \mathbb{N}^{|I|}$  is a multi index and  $\mu, \nu \in \mathbb{N}^{4|I|}$  are multi quadri indices. Therefore the term  $\partial^\mu \delta(I - x) = \prod_{i \in I} \partial^{\mu_i} \delta(y_i - x)$  (and  $\mu_1$  can be  $\mu\nu\rho$  for example, with  $\mu, \nu, \rho$  usual Lorentz indices). The number  $\omega_I$  refers to the singular order of the numerical distribution  $\omega_0(T(I, \varphi^2)(y_I, x))$  and is given by  $\omega_I = \sum_{i \in I} \dim W_i + 2 \dim \varphi - 4|I|$ . Hence the sum over  $I$  only runs over subsets for which the coincidence of  $x_1, x_2$  produces distributions with non negative singular order. These are ordered into subgraphs by the sum over  $\gamma$ . Only graphs for which their corresponding order, namely  $\omega_I - |\gamma|$ , is non negative ( $\gamma$  is a derivative w.r.t.  $\varphi$  and therefore the singular order decreases with increasing  $|\gamma|$ ) are taken into account. The sum over  $\alpha$  refers to the usual subtraction procedure (running from 0 up to the order of singularity of the corresponding distribution). As a matter of fact it has to be split into an action on  $\delta$  and on  $T(\dots, \varphi^\gamma)$  since we changed the coordinate of  $\varphi^{\gamma_i}(y_i)$  into  $\varphi^{\gamma_i}(x)$  according to the  $\delta$  function (see also the example at the beginning).

To explain the coefficients, we have to introduce the corresponding  ${}^0T$ -product which is the same expression like (6.15) up to the last sum which does not contain the term  $I = N$ . So

$$\begin{aligned} {}^0T(N, : \varphi, \varphi :)(y_N, x_1, x_2) = & \\ & = T(N, \varphi, \varphi)(y_N, x_1, x_2) - \omega_0(T(\varphi, \varphi)(x_1, x_2)) T(N)(y_N) + \\ & - \sum_{\substack{I \subset N \\ I \neq \emptyset \\ I \neq N}} \sum_{|\gamma| \leq \omega_I} \sum_{|\alpha| \leq \omega_I - |\gamma|} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \\ & \times (e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi)) T(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x). \end{aligned} \quad (6.16)$$

Hence  $T$  and  ${}^0T$  only differ by a term with support on  $\text{Diag}_n^x$ . The coefficients  $e_I^{\alpha(\gamma)}$  are given by the expression

$$e_I^{\alpha(\gamma)}(\xi) = \int dz_I z^\alpha \omega_0 \left( {}^0T \left( I^{(\gamma)}, : \varphi, \varphi : \right) (y_I, \xi, -\xi) \right) w_{I^{(\gamma)}}(z_I), \quad (6.17)$$

where again  $I^{(\gamma)} = \{W_i^{(\gamma_i)}, i \in I\}$  and the exponent  $(\gamma_i)$  means  $\gamma_i$  fold differentiation with respect to  $\varphi$  in  $\mathfrak{B}$ . The coefficients  $a_I^{\alpha(\gamma)}$  are chosen in such a way, that Lorentz covariance is conserved. The function  $w_{I^{(\gamma)}}$  is the auxiliary function used in the extension process of the distribution  $\omega_0(T(I^{(\gamma)}, \varphi^2)(y_I, x))$  of the  $|I|$  difference coordinates  $y_1 - x, \dots, y_{|I|} - x$ .<sup>1</sup>

We require our bilocal products to fulfil normalization condition **N3** in the corresponding form, namely

$$\begin{aligned} [T(N, : \varphi, \varphi :)(y_N, x_1, x_2), \varphi(z)] = & \\ = i \sum_{k=1}^n T(N^{(e_k)}, : \varphi, \varphi :)(y_N, x_1, x_2) \Delta(y_k - z) + & \\ + iT(N, \varphi)(y_N, x_1) \Delta(x_2 - z) + iT(N, \varphi)(y_N, x_2) \Delta(x_1 - z), & \end{aligned} \quad (\text{N3}')$$

<sup>1</sup>Without loss of generality  $I = \{1, \dots, |I|\}$ .

and  $e_k = (0, \dots, 1, \dots, 0) \in \mathbb{N}^n$  with the 1 in the  $k$ 'th position.

**1.3. Proof of the restriction property.** We show that (6.15) yields the right  $T$ -product by restricting  $x_1 = x_2$ . Our proof requires the validity of **N3'** which is proven in the next subsection. Unfortunately our proof still lacks an existence statement for the distributions  $a_I(\xi)$ , necessary for the conservation of Poincaré covariance. Hence we have to assume that they exist.

Assume that up to order  $n - 1$  the bilocal  $T(N, : \varphi, \varphi :)(y_N, x_1, x_2)$  exists and its restriction  $x_1 = x_2$  is given by  $T(N, \varphi^2)(y_N, x)$ . We show that  $T$  has the right causal factorization (6.13). We write (6.15) as

$$\begin{aligned} T(N, : \varphi, \varphi :)(y_N, x_1, x_2) &= T(N, \varphi, \varphi)(y_N, x_1, x_2) + \\ &- \sum_{I \subset N} \sum_{|\gamma| \leq \omega_I} \sum_{|\mu| + |\nu| \leq \omega_I - |\gamma|} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) T(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x). \end{aligned} \quad (6.18)$$

The  $E^{\mu\nu(\gamma)}$  are given by

$$E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) = \frac{(-)^\mu}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) (e_I^{\mu+\nu(\gamma)}(\xi) - a_I^{\mu+\nu(\gamma)}(\xi)), \quad (6.19)$$

$$\text{with } E^{\mu\nu(\gamma)}(\emptyset, \varphi, \varphi)(x, \xi) = \delta_0^\mu \delta_0^\nu \delta_0^\gamma \omega_0(T(\varphi, \varphi)(x_1, x_2)) \big|_{x_1 - x_2 = 2\xi} \quad (6.20)$$

$$\text{and } \text{supp } E^{\mu\nu(\gamma)}(N, \varphi, \varphi)(y_N, x, \xi) \subset \text{Diag}_n^x, N \neq \emptyset. \quad (6.21)$$

If  $L \gtrsim N \setminus L, x, x_1, x_2$ , we have from (6.18):

$$\begin{aligned} T(N, : \varphi, \varphi :)(y_N, x_1, x_2) &= \\ &= T(L)(y_L) T(N \setminus L, \varphi, \varphi)(y_{N \setminus L}, x_1, x_2) + \\ &- T(L)(y_L) E(\emptyset, \varphi, \varphi)(x, \xi) T(N \setminus L)(y_{N \setminus L}) + \\ &- \sum_{\substack{I \subset N \setminus L \\ I \neq \emptyset}} \sum_{\gamma, \mu, \nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) T(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x) \\ &= T(L)(y_L) T(N \setminus L, \varphi, \varphi)(y_N, x_1, x_2) + \\ &- T(L)(y_L) \sum_{I \subset N \setminus L} \sum_{\gamma, \mu, \nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) \times \\ &\quad \times T((N \setminus L) \setminus I, \partial^\nu \varphi^\gamma)(y_{(N \setminus L) \setminus I}, x) \\ &= T(L)(y_L) T(N \setminus L, : \varphi, \varphi :)(y_{N \setminus L}, x_1, x_2). \end{aligned} \quad (6.22)$$

If  $L, x_1 \gtrsim N \setminus L, x_2, x$ , with  $x_1 \gtrsim x_2$  we find:

$$\begin{aligned}
T(N, : \varphi, \varphi :)(y_N, x_1, x_2) &= \\
&= T(L, \varphi)(y_L, x_1)T(N \setminus L, \varphi)(y_{N \setminus L}, x_2) + \\
&\quad - T(L)(y_L)E(\emptyset, \varphi, \varphi)(x, \xi)T(N \setminus L)(y_{N \setminus L}) + \\
&\quad - \sum_{\substack{I \subset N \setminus L \\ I \neq \emptyset}} \sum_{\gamma, \mu, \nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi)T(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x) + \\
&= T(L, \varphi)(y_L, x_1)T(N \setminus L, \varphi)(y_{N \setminus L}, x_2) + \\
&\quad - T(L)(y_L) \sum_{I \subset N \setminus L} \sum_{\gamma, \mu, \nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) \times \\
&\quad \times T((N \setminus L) \setminus I, \partial^\nu \varphi^\gamma)(y_{(N \setminus L) \setminus I}, x) \\
&= T(L, \varphi)(y_L, x_1)T(N \setminus L, \varphi)(y_{N \setminus L}, x_2) - T(L)(y_L) \times \\
&\quad \times (T(N \setminus L, \varphi, \varphi)(y_{N \setminus L}, x_1, x_2) - T(N \setminus L, : \varphi, \varphi :)(y_{N \setminus L}, x_1, x_2)).
\end{aligned} \tag{6.23}$$

A similar calculation also shows the right causality decomposition, if  $x$  is in the later set.

Because of this causal factorization property and the inductive assumption we immediately have

$${}^0T(N, : \varphi, \varphi :)(y_N, x, x) = {}^0T(N, \varphi^2)(y_N, x), \quad (y_N, x) \in \mathbb{M}^{n+1} \setminus \text{Diag}_{n+1}. \tag{6.24}$$

We show that the same equation also holds for the  $T$ -products. Because of **N3'** it is sufficient to consider vacuum expectation values only. Inserting the definition (6.15) we find:

$$\begin{aligned}
\omega_0(T(N, : \varphi, \varphi :)(y_N, x_1, x_2)) - \omega_0({}^0T(N, : \varphi, \varphi :)(y_N, x_1, x_2)) &= \\
&= - \sum_{|\alpha| \leq \omega_N} \frac{(-)^{|\alpha|}}{\alpha!} \partial^\alpha \delta(y_N - x) (e_N^{\alpha(0)}(\xi) - a_N^{\alpha(0)}(\xi)).
\end{aligned} \tag{6.25}$$

Since the vacuum expectation values are translation invariant we use the coordinates  $z_i = y_i - x$  and  $z = (z_1, \dots, z_n)$ . We set

$$t(z, \xi) \doteq \omega_0(T(N, : \varphi, \varphi :)(z, \xi, -\xi)), \tag{6.26}$$

$${}^0t(z, \xi) \doteq \omega_0({}^0T(N, : \varphi, \varphi :)(z, \xi, -\xi)). \tag{6.27}$$

It follows that

$$e_N^{\alpha(0)}(\xi) = \int dz_N {}^0t(z, \xi) z^\alpha w_{N(0)}(z) \tag{6.28}$$

in this notation. Now, by smearing with  $f \in \mathcal{D}(\mathbb{M}^n)$  in the  $z$  coordinates we find:

$$\langle t(\cdot, \xi), f \rangle = \langle {}^0t(\cdot, \xi), W_{(\omega_N; w_{N(0)})} f \rangle + \sum_{|\alpha| \leq \omega_N} \frac{a_N^{\alpha(0)}(\xi)}{\alpha!} \partial^\alpha f(0). \tag{6.29}$$

Because of the sufficient subtraction on the test function we can put  $\xi = 0$ . Then we have

$$t(z, 0) = \omega_0 (T(N, \varphi^2)(z_N, 0)), \quad (6.30)$$

where the constants  $a_N^{\alpha(0)}(0)$  can be chosen to produce any normalization of the RHS.

**1.4. Lorentz covariance.** Due to the definition the bilocal  $T$ -products are translation covariant. Namely, they are products of translation invariant numerical distributions and operator valued Wick products because of **N3'**. Therefore we have to consider Lorentz covariance for the the numerical distributions only.

If  ${}^0t$  transforms covariantly under the Lorentz group

$$\Lambda^0 t(z, \xi) = D(\Lambda^{-1})^0 t(z, \xi), \quad (6.31)$$

w.r.t. both variables we find that  $t$  transforms the same way, if<sup>2</sup>

$$(\Lambda^\alpha_\beta D(\Lambda)\Lambda - \delta^\alpha_\beta) e^\beta(\xi) = (\Lambda^\alpha_\beta D(\Lambda)\Lambda - \delta^\alpha_\beta) a^\beta(\xi) \quad (6.32)$$

w.r.t.  $\xi$ . With  $e^\beta$  from (6.17) this leads to

$$\int dz {}^0t(z, \xi) z^\alpha (\Lambda w - w)(z) = (\Lambda^\alpha_\beta D(\Lambda)\Lambda - \delta^\alpha_\beta) a^\beta(\xi). \quad (6.33)$$

The RHS is a one coboundary, which we have to solve for  $a^\beta$ . We assume that there are solutions  $\mathcal{D}'(\mathbb{M}) \ni a^\beta \neq e^\beta$  with the property that  $a(0)$  exists. Unlike in the case of usual  $T$ -products (cf. chapter 3, section 5) we have no existence proof, so we have to impose it as an assumption. Moreover we see that  $a^\beta(\xi)$  is determined by the RHS of (6.33) only up to terms  $h^\beta(\xi)$ , with  $\Lambda^\alpha_\beta D(\Lambda)\Lambda h^\beta(\xi) = h^\alpha(\xi)$ .

If the central solution ( $w = 1$ ) exists  $a$  can be chosen as

$$a^\alpha(\xi) = \int dz {}^0t(z, \xi) z^\alpha (w - 1)(z), \quad (6.34)$$

which fulfils all properties. It simply replaces the subtraction with auxiliary function  $w$  by the central subtraction.

**1.5. Proof of N3'.** We show that **N3'** holds by evaluating both sides independently. In the calculation we use the following formula, taken from [DF00b]:

$$\frac{1}{\alpha!} \frac{\partial(\partial^\alpha V)}{\partial \varphi_r} = \sum_{\mu+\nu=\alpha} \frac{1}{\mu! \nu!} \partial^\mu \left( \frac{\partial V}{\partial \varphi} \right) \delta_r^\nu, \quad (6.35)$$

where  $V \in \mathfrak{B}$  is supposed to contain no derivated fields. We first investigate the contribution to the commutator of the RHS of **N3'** arising from the sum in

---

<sup>2</sup>We suppress the indices  $I$  and  $\gamma$  which are fixed in this problem.

(6.15). The following term appears:

$$\begin{aligned}
& \frac{1}{\nu_1! \dots \nu_{|I|}! \gamma_1! \dots \gamma_{|I|}!} [T(N \setminus I, \partial^{\nu_1} \varphi^{\gamma_1} \dots \partial^{\nu_{|I|}} \varphi^{\gamma_{|I|}})(y_N, x), \varphi(z)] = \\
& = \sum_{k \in N \setminus I} \frac{1}{\nu! \gamma!} T\left((N \setminus I)^{(e_k)}, \partial^\nu \varphi^\gamma\right)(y_{N \setminus I}, x) i\Delta(y_k - x) + \\
& + \sum_{k \in I} \sum_{\sigma_k + \rho_k = \nu_k} \frac{1}{\nu_1! \gamma_1! \dots \nu_{|I|}! \gamma_{|I|}!} \frac{1}{\sigma_k! \rho_k! (\gamma_k - 1)!} \times \\
& \times T(N \setminus I, \partial^{\nu_1} \varphi^{\gamma_1} \dots \partial^{\sigma_k} \varphi^{\gamma_k - 1} \dots \partial^{\nu_{|I|}} \varphi^{\gamma_{|I|}})(y_N, x) i\partial^{\rho_k} \Delta(x - z). \tag{6.36}
\end{aligned}$$

Without loss of generality we have put  $I = \{1, \dots, |I|\}$ . Denote the  $T$ -product on the last line symbolically by  $T^{\sigma_k}$ . Note that it only appears, if  $|\gamma| > 0$ . With the equality

$$\frac{(-)^{|\beta|}}{\beta!} \partial^\beta \delta(y - x) f(y) = \sum_{\rho + \sigma = \beta} \frac{(-)^{|\rho|}}{\rho! \sigma!} \partial^\rho \delta(y - x) \partial^\sigma f(x), \tag{6.37}$$

we have for every  $k \in I$  (now  $I$  can be any subset of  $N$ ) the following contribution:

$$\begin{aligned}
& \sum_{\mu_k + \nu_k = \alpha_k} \frac{(-)^{|\mu_k|}}{\mu_k!} \partial^{\mu_k} \delta(y_k - x) \sum_{\sigma_k + \rho_k = \nu_k} \frac{1}{\sigma_k! \rho_k!} T^{\sigma_k} \partial^{\rho_k} \Delta(x - z) = \\
& \sum_{\mu_k + \nu_k = \alpha_k} \frac{(-)^{|\mu_k|}}{\mu_k! \nu_k!} \partial^{\mu_k} \delta(y_k - x) T^{\nu_k} \Delta(y_k - z). \tag{6.38}
\end{aligned}$$

Now we insert this result into the last term of (6.15), commuted with  $\varphi(z)$ :

$$\begin{aligned}
& [\text{last term of (6.15)}, \varphi(z)] = \\
& = \sum_{\substack{I \subset N \\ I \neq \emptyset}} \sum_{|\gamma| \leq \omega_I} \sum_{|\alpha| \leq \omega_I - |\gamma|} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \\
& \times (e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi)) \sum_{k \in N \setminus I} T\left(N^{(e_k)} \setminus I, \partial^\nu \varphi^\gamma\right)(y_{N \setminus I}, x) i\Delta(y_k - x) + \\
& + \sum_{\substack{I \subset N \\ I \neq \emptyset}} \sum_{1 \leq |\gamma| \leq \omega_I} \sum_{k \in I} \sum_{|\alpha| \leq \omega_I - |\gamma|} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! (\gamma - e_k)!} \partial^\mu \delta(I - x) \times \\
& \times (e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi)) T(N \setminus I, \partial^\nu \varphi^{\gamma - e_k})(y_{N \setminus I}, x) i\Delta(y_k - x). \tag{6.39}
\end{aligned}$$

If we shift the multi index  $\gamma \rightarrow \gamma + e_k$  in the second term and note that  $e_I^{\alpha(\gamma+e_k)} = e_{I(e_k)}^{\alpha(\gamma)}$  by definition of  $a_I^{\alpha(\gamma)}$ , (6.17), we have:

last term of (6.39) =

$$= \sum_{\substack{I \subset N \\ I \neq \emptyset}} \sum_{0 \leq |\gamma| \leq \omega_I - 1} \sum_{k \in I} \sum_{|\alpha| \leq \omega_I - |\gamma| - 1} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \quad (6.40)$$

$$\times (e_{I(e_k)}^{\alpha(\gamma)}(\xi) - a_{I(e_k)}^{\alpha(\gamma)}(\xi)) T(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x) i\Delta(y_k - x).$$

Then we can sum up both terms and obtain:

$$[\text{last term of (6.15)}, \varphi(z)] =$$

$$= \sum_{k=1}^n \sum_{\substack{I \subset N^{(e_k)} \\ I \neq \emptyset}} \sum_{|\gamma| \leq \omega_I} \sum_{|\alpha| \leq \omega_I - |\gamma|} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \quad (6.41)$$

$$\times (e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi)) T(N^{(e_k)} \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x) i\Delta(y_k - x).$$

Note, that if  $k \in I$  the singular order  $\omega_I$  in (6.41) is automatically lowered by one compared to (6.40) since now  $k \in N^{(e_k)}$ . In (6.40) the singular order is measured with respect to the set  $I \subset N$  without differentiation of the  $k$ 'th symbol.

This shows the equality of the last term, if we insert (6.15) into **N3'**. The remaining two terms of (6.15) on the LHS of **N3'** commute with  $\varphi$  according to

$$[T(N, \varphi, \varphi)(y_N, x_1, x_2) - \omega_0(T(\varphi, \varphi)(x_1, x_2))T(N)(y_N), \varphi(z)] =$$

$$= \sum_{k=1}^n \left[ T(N^{(e_k)}, \varphi, \varphi)(y_N, x_1, x_2) + \right. \quad (6.42)$$

$$\left. - \omega_0(T(\varphi, \varphi)(x_1, x_2))T(N^{(e_k)})(y_N) \right] i\Delta(y_k - z) +$$

$$+ T(N, \varphi)(y_N, x_1) i\Delta(x_2 - z) + T(N, \varphi)(y_N, x_2) i\Delta(x_1 - z),$$

such that they match the missing terms in the RHS of **N3'**. This finishes the proof.

## 2. The operator product expansion

In order to derive the interacting normal product  $(:\varphi, \varphi:)_g$  we only have to evaluate the  $R$  products which are given in terms of  $T$ -products according to



(4.12).

$$\begin{aligned}
R(N; \varphi, \varphi)(y_N, x_1, x_2) - R(N; : \varphi, \varphi :)(y_N, x_1, x_2) &= \\
&= \sum_{I \subset N} (-)^{|I|} \bar{T}(I)(y_I) [T(N \setminus I, \varphi, \varphi)(y_{N \setminus I}, x_1, x_2) + \\
&\quad - T(N \setminus I, : \varphi, \varphi :)(y_{N \setminus I}, x_1, x_2)] \\
&= \sum_{I \sqcup J \sqcup K = N} \sum_{\mu, \nu, \gamma} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) (-)^{|J|} \bar{T}(J)(y_J) T(K, \partial^\nu \varphi^\gamma)(y_K, x) \\
&= \sum_{I \subset N} \sum_{\mu, \nu, \gamma} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) R(N \setminus I, \partial^\nu \varphi^\gamma)(y_{N \setminus I}, x).
\end{aligned} \tag{6.43}$$

If we now insert into the power series for the interacting fields (4.11) only the  $\mu = 0$  coefficients contribute, since  $g \upharpoonright_{\mathcal{O}} = \text{const.}$  We omit this index on the  $E^{\mu\nu(\gamma)}$ -terms. The RHS of (6.43) is just the  $n$ 'th order contribution of the product of two power series. So we find the expansion:

$$T(\varphi, \varphi)_{g\mathcal{L}}(x_1, x_2) = (: \varphi, \varphi : )_{g\mathcal{L}}(x_1, x_2) + \sum_{|\gamma| \leq 2} \sum_{\alpha \leq 2 - |\gamma|} E_{g\mathcal{L}}^{\alpha(\gamma)}(\xi) (\partial^\alpha \varphi^\gamma)_{g\mathcal{L}}(x). \tag{6.44}$$

This is the *operator product expansion*. We used the fact, that the maximal singular order  $\omega_0(T(N, \varphi^2))$  is two in a renormalizable field theory. If the interaction is a sum of Wick monomials  $g \cdot \mathcal{L} = \sum_{r=1}^s g_r \mathcal{L}_r$ , the expansion coefficients read:

$$\begin{aligned}
E_{g\mathcal{L}}^{\alpha(\gamma)}(\xi) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{r_1, \dots, r_n=1}^s g_{r_1}(x) \dots g_{r_n}(x) \times \\
&\quad \times \sum_{|\gamma| \leq \omega_{r_1, \dots, r_n}} \sum_{|\alpha| \leq \omega_{r_1, \dots, r_n} - |\gamma|} \left[ -a_{r_1, \dots, r_n}^{\alpha(\gamma)}(\xi) + \right. \\
&\quad \left. + \int dz_N z^\alpha \omega_0 \left( {}^0T \left( \mathcal{L}_{r_1}^{(\gamma_1)}, \dots, \mathcal{L}_{r_n}^{(\gamma_n)}, : \varphi, \varphi : \right) (z_N, \xi, -\xi) \right) w_{r_1 \dots r_n}^{\gamma_1 \dots \gamma_n}(z_N) \right].
\end{aligned} \tag{6.45}$$

As was expected from the general theorem of the unitary equivalence of the local algebras [BF96], the expansion coefficients depend only locally on  $g$ .

The operator product expansion (6.44) has the general form:

$$\begin{aligned}
T(\varphi, \varphi)_{g\mathcal{L}}(x_1, x_2) &= (: \varphi, \varphi : )_{g\mathcal{L}}(x_1, x_2) + E_{g\mathcal{L}}^{0(0)}(\xi) \mathbb{I}_{g\mathcal{L}} + E_{g\mathcal{L}}^{0(1)}(\xi) \varphi_{g\mathcal{L}}(x) + \\
&\quad + E_{g\mathcal{L}}^{\mu(1)}(\xi) \partial_\mu \varphi_{g\mathcal{L}}(x) + E_{g\mathcal{L}}^{0(2)}(\xi) (\varphi^2)_{g\mathcal{L}}(x).
\end{aligned} \tag{6.46}$$

If we consider a pure  $\varphi^4$ -coupling we have  $E_{g\mathcal{L}}^{0(1)} = E_{g\mathcal{L}}^{\mu(1)} = 0$ , since the vacuum expectation value of an odd number of fields is zero. In that case the coefficients

read

$$E_{g\mathcal{L}}^{0(0)}(\xi) = i\Delta_F(2\xi) + \sum_{n=2}^{\infty} \frac{i^n}{n!} g(x)^n \left[ -a_{N^{(0)}}^{0(0)}(\xi) + \int dz_N \omega_0 \left( {}^0T(\mathcal{L}, \dots, \mathcal{L}, : \varphi, \varphi:) (z_N, \xi, -\xi) \right) w_{N^{(0)}}(z_N) \right] \quad (6.47)$$

$$E_{g\mathcal{L}}^{0(2)}(\xi) = \sum_{n=1}^{\infty} \frac{i^n}{n!} g(x)^n \sum_{|\gamma|=2} \frac{1}{\gamma_1! \dots \gamma_n!} \left[ -a_{N^{(\gamma)}}^0(\xi) + \int dz_N \omega_0 \left( {}^0T(\mathcal{L}^{(\gamma_1)}, \dots, \mathcal{L}^{(\gamma_n)}, : \varphi, \varphi:) (z_N, \xi, -\xi) \right) w_{N^{(\gamma)}}(z_N) \right]. \quad (6.48)$$

A closer inspection of the two terms reveals that  $E_{g\mathcal{L}}^{0(0)}$  contains the terms which appear in the mass and wave function renormalization. If we put  $w = 1$  (which is allowed if  $m > 0$ ) and do a resummation over one particle irreducible graphs, we would end up with the usual geometric series found in the literature for the interacting propagator (cf. [IZ85]). In that case all disconnected graphs disappear. But this is only due to that special choice of normalization. The series  $E_{g\mathcal{L}}^{0(2)}$  contain the contributions to the renormalization of the coupling constant.

### 3. Towards the definition of a state

In this section we introduce the idea to define a state on the algebra of local observables with the help of the OPE. We remind the definition of a state  $\omega$  as a linear normed positive functional in the free field theory. In that case there is also an OPE, namely Wick's theorem (cf. (3.19) – (3.21)). The vacuum state  $\omega_0$  on the algebra of observables (of the free field) was defined by  $\omega_0(:A:) = 0$  and  $\omega_0(\mathbb{I}) = 1$ . Since Wick's theorem allows to expand any observable into a series of Wick polynomials the state is uniquely defined.

In the interacting field theory the fields are (operator valued distributional) formal power series in  $g$ . Denote by  $\mathbb{C}_g$  the formal power series in  $g$  with complex coefficients. Following [DF99] a state  $\omega_{g\mathcal{L}}$  on  $\mathfrak{A}_{g\mathcal{L}}(\mathcal{O})$  is a mapping:

$$\omega_{g\mathcal{L}} : \mathfrak{A}_{g\mathcal{L}}(\mathcal{O}) \mapsto \mathbb{C}_g, \quad (6.49)$$

$$\omega_{g\mathcal{L}}(a_g A_{g\mathcal{L}} + B_{g\mathcal{L}}) = a_g \omega_{g\mathcal{L}}(A_{g\mathcal{L}}) + \omega_{g\mathcal{L}}(B_{g\mathcal{L}}), \quad a_g \in \mathbb{C}_g \quad (6.50)$$

$$\omega_{g\mathcal{L}}(A_{g\mathcal{L}}^*) = \overline{\omega_{g\mathcal{L}}(A_{g\mathcal{L}})} \quad (6.51)$$

$$\omega_{g\mathcal{L}}(\mathbb{I}_{g\mathcal{L}}) = 1, \quad (6.52)$$

$$\omega_{g\mathcal{L}}(A_{g\mathcal{L}}^* A_{g\mathcal{L}}) \geq 0, \quad (6.53)$$

$A_{g\mathcal{L}}, B_{g\mathcal{L}} \in \mathfrak{A}_{g\mathcal{L}}(\mathcal{O})$ . Inspired by the definition of the vacuum state for the free field algebra we set

$$\omega_{g\mathcal{L}}((:A:)_{g\mathcal{L}}) = 0, \quad (6.54)$$

$$\omega_{g\mathcal{L}}(\mathbb{I}_{g\mathcal{L}}) = 1. \quad (6.55)$$

Unfortunately we do not have an OPE for the general product of interacting fields and time ordered products of them. But we can already check, if the above criteria hold in our case.

We consider pure  $\varphi^4$ -interaction. Since  $(:\varphi, \varphi:)^*_{g\mathcal{L}} = (:\varphi, \varphi:)_{g\mathcal{L}}$  on  $\mathcal{D}$  the adjoint of (6.46) is:

$$\overline{T}(\varphi, \varphi)_{g\mathcal{L}}(x_1, x_2) = (:\varphi, \varphi:)_{g\mathcal{L}}(x_1, x_2) + \overline{E_{g\mathcal{L}}^{0(0)}(\xi)} \mathbb{I}_{g\mathcal{L}} + \overline{E_{g\mathcal{L}}^{0(2)}(\xi)} (\varphi^2)_{g\mathcal{L}}(x). \quad (6.56)$$

The singular order of  $E^{0(0)}$  is 2 and of  $E^{0(2)}$  is 0. Therefore we can form

$$\begin{aligned} \varphi_{g\mathcal{L}}(x_1)\varphi_{g\mathcal{L}}(x_2) &= \\ &= \theta(x_1^0 - x_2^0)T(\varphi, \varphi)_{g\mathcal{L}}(x_1, x_2) + \theta(x_2^0 - x_1^0)\overline{T}(\varphi, \varphi)_{g\mathcal{L}}(x_2, x_1). \end{aligned} \quad (6.57)$$

Then the interacting two point function is given by

$$\omega_{2g\mathcal{L}}(x_1, x_2) \doteq \omega_{g\mathcal{L}}(\varphi_{g\mathcal{L}}(x_1)\varphi_{g\mathcal{L}}(x_2)) = \theta(\xi^0)E_{g\mathcal{L}}^{0(0)}(\xi) + \theta(-\xi^0)\overline{E_{g\mathcal{L}}^{0(0)}(\xi)}. \quad (6.58)$$

For the notion of positivity we refer to the work [DF99]. A formal power series  $\mathbb{C}_g \ni b_g = \sum_n b_n g^n$  is defined to be positive if it can be written as the square of another power series  $b_g = \overline{c_g}c_g$ . This is equivalent to the conditions:  $b_n \in \mathbb{R}, \forall n \in \mathbb{N}_0$  and for the first non vanishing  $b_l$  it is required that  $b_l > 0$  and  $l$  is even. So for  $f \in \mathcal{D}(\mathbb{M})$  we have:

$$\begin{aligned} \omega_{2,g\mathcal{L}}(\overline{f}, f) &= \\ &= \int dx dy \left( \theta(x^0 - y^0)E_{g\mathcal{L}}^{0(0)}(x - y) + \theta(y^0 - x^0)\overline{E_{g\mathcal{L}}^{0(0)}(x - y)} \right) \overline{f(x)}f(y) \\ &= 2 \int dx dy \operatorname{Re} \theta(x^0 - y^0)E_{g\mathcal{L}}^{0(0)}(x - y)\overline{f(x)}f(y), \end{aligned} \quad (6.59)$$

where we exchanged  $x \leftrightarrow y$  in the second integral and used the fact that  $E_{g\mathcal{L}}^{0(0)}$  is even. The first non vanishing contribution comes from the free two point function which is positive, cf. (2.22). Under the above criterion our two point function is positive.

#### 4. Remarks

Our OPE is a consequence of the definition of our normal product of two fields. The definition for other but scalar fields can be derived straight forward by the applied methods. A generalization to composed fields requires a modification of the terms arising by  $x_1, x_2$ -contractions. Unfortunately, we have not succeeded in finding a definition for higher order ( $> \text{bi}$ ) normal products.

On the other hand the bi-OPE already contains all contributions which are necessary for the definition of a mass-, wavefunction- and coupling normalization, which up to now always requires the adiabatic limit [Sch95], [EG73]. For

the first two of them this should be possible by a suitable condition on the measure that one can derive from the Jost-Lehmann-Dyson representation of our (time ordered) two point function.

## CHAPTER 7

### Conclusion and Outlook

We have provided for an explicit Poincaré covariant normalization in the Epstein-Glaser approach to renormalization theory. Beside its meaning as a closed loophole in the inductive construction it can be useful for the practitioner. Especially for massless theories where the central solution does not exist this is an advantage.

The local conservation of translation invariance by means of conservation of the EMT is likely to be generalized to an arbitrary coupling containing also derivated fields by the same method. Discussing these (and all other) symmetries locally by suitable Ward identities moreover shows the advantage that massive and massless fields can be treated on the same footing.

An open problem still is a local definition of the  $\beta$ -function. The validity of the equation  $\Theta_{\text{imp } g\mathcal{L}}^\mu = 2\beta\mathcal{L}_{g\mathcal{L}}$ , as conjectured by Minkowski [Min76] and verified by Zimmermann's normal product quantization [Zim84] has to be examined in the local perturbative approach. This might reveal how the free parameter which is still present in the trace anomaly in  $\varphi^4$ -theory as discussed by us has to be chosen. A similar problem addresses the anomalous dimension. The anomalous terms found by our procedure still lack the possibility to derive the anomalous dimension of the interacting field as a real parameter (in form of a formal power series).

The OPE has produced power series as algebraic structure constants. These depend on the coupling only locally and may therefore serve as the right objects for the discussion of wave function, mass and coupling constant renormalization independent of the adiabatic limit.

*“... There is another theory which states that this has already happened.”*

– Douglas Adams, "The Restaurant at the End of the Universe"

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## APPENDIX A

### Representation of the symmetric groups

Everything in this brief appendix should be found in any book about representation theory of finite groups. We refer to [Sim96, Boe70, FH91]. The group algebra  $\mathcal{A}_{S_p}$  consists of elements

$$a = \sum_{g \in S_p} \alpha(g) \cdot g, \quad b = \sum_{g \in S_p} \beta(g) \cdot g, \quad (\text{A.1})$$

where  $\alpha, \beta$  are arbitrary complex numbers. The sum of two elements is naturally given by the summation in  $\mathbb{C}$  and the product is defined through the following convolution:

$$ab \doteq \sum_{g_1, g_2} \alpha(g_1) \beta(g_2) \cdot g_1 g_2 = \sum_g \gamma(g) \cdot g \text{ with} \quad (\text{A.2})$$

$$\gamma(g) \doteq \sum_{g_1 g_2 = g} \alpha(g_1) \beta(g_2) = \sum_{g_1} \alpha(g_1) \beta(g_1^{-1} g) = \sum_{g_2} \alpha(g g_2^{-1}) \beta(g_2). \quad (\text{A.3})$$

The group algebra is the direct sum of simple twosided ideals:

$$\mathcal{A}_{S_p} = I_1 \oplus \cdots \oplus I_k, \quad (\text{A.4})$$

and  $k$  is the number of partitions of  $p$ . Every ideal  $I_j$  contains  $f_j$  equivalent irreducible representations of  $S_p$ .  $I_j$  is generated by an idempotent  $e_j \in \mathcal{A}_{S_p}$ :

$$I_j = \mathcal{A}_{S_p} e_j \quad e_j^2 = e_j. \quad (\text{A.5})$$

These idempotents satisfy the following orthogonality and completeness relations:

$$e_j e_i = \delta_{ji} \quad \sum_{j=1}^k e_j = \mathbb{I}. \quad (\text{A.6})$$

The center of  $\mathcal{A}_{S_p}$  consists of all elements  $\sum_j^k \alpha_j e_j, \alpha_j \in \mathbb{C}$ .

Every permutation of  $S_p$  can be uniquely (modulo order) written as a product of disjoint cycles. Since two cycles are conjugated if and only if their length is the same, the number of conjugacy classes is equal to the number of partitions of  $p$ . Denoting the  $j$ 'th conjugacy class by  $c_j$  we build the sum of all elements of one class

$$k_j \doteq \sum_{\pi \in c_j} \pi \in \mathcal{A}_{S_p} \quad (\text{A.7})$$

which is obviously in the center of  $\mathcal{A}_{S_p}$ , too. So we can expand  $k_i$  in the basis  $e_j$ :

$$k_i = h_i \sum_{j=1}^k \frac{1}{f_j} \chi_j(c_i) e_j, \quad (\text{A.8})$$

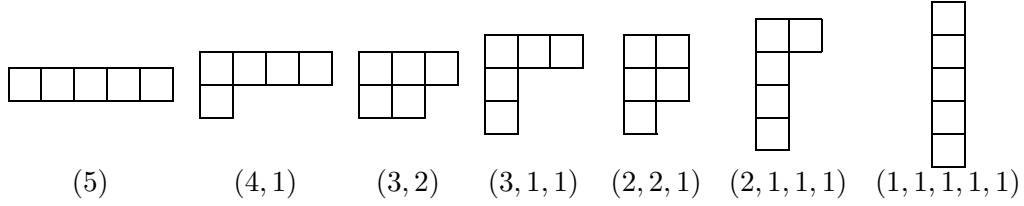
where  $\chi_j(c_i)$  is the character of the class  $c_i$  in the representation generated by  $e_j$  and  $h_i$  is the number of elements of  $c_i$ . The dimension of that representation is equal to the multiplicity  $f_j$ .

The construction of the idempotents can be carried out via the

**Young tableaux.** A sequence of integers  $(m) = (m_1, \dots, m_r), m_1 \geq m_2 \geq \dots \geq m_r$  with  $\sum_{j=1}^r m_j = p$  gives a partition of  $p$ . To every such sequence we associate a diagram with

$$\begin{array}{l} m_1 \text{ boxes} \\ m_2 \text{ boxes} \\ \vdots \end{array} \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \dots \begin{array}{|c|} \hline \\ \hline \end{array}$$

called a *Young frame*  $(m)$ . Let us take  $p = 5$  as an example:



An assignment of numbers  $1, \dots, p$  into the boxes of a frame is called a *Young tableau*. Given a tableau  $T$ , we denote  $(m)$  by  $(m)(T)$ . If the numbers in every row and in every column increase the tableau is called *standard*. The number of standard tableaux for the frame  $(m)$  is denoted by  $f_{(m)}$ . It is equal to the dimension of the irreducible representation generated by the idempotent  $e_{(m)}$ . We now answer the question

*How to construct  $e_{(m)}$ .* Set

$$\begin{aligned} \mathcal{R}(T) &\doteq \{\pi \in S_p | \pi \text{ leaves each row of } T \text{ set wise fixed}\}, \\ \mathcal{C}(T) &\doteq \{\pi \in S_p | \pi \text{ leaves each column of } T \text{ set wise fixed}\}, \end{aligned}$$

and build the following objects:

$$P(T) \doteq \sum_{p \in \mathcal{R}(T)} p, \quad Q(T) \doteq \sum_{q \in \mathcal{C}(T)} \text{sgn}(q) q, \quad (\text{A.9})$$

then

$$e(T) \doteq \frac{f_{(m)}}{p!} P(T) Q(T) \quad (\text{A.10})$$



is a minimal projection in  $\mathcal{A}_{S_p}$  (generates a minimal left ideal). The central projection (generating the simple two sided ideal) is given by

$$e_{(m)} \doteq \frac{f_{(m)}}{p!} \sum_{T|(m)(T)=(m)} e(T). \quad (\text{A.11})$$

**Example**  $p = 3$ . The frame  $\square\square\square$  has only one standard tableau  $\begin{smallmatrix} \boxed{1} & \boxed{2} & \boxed{3} \end{smallmatrix}$ . All different tableaux in (A.11) lead to the same idempotent (A.10) which is just the sum of all permutations.

$$e_{(3)} = \frac{1}{6}(\mathbb{I} + (12) + (13) + (23) + (123) + (132)).$$

For the frame  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$  we only need the column permutations in (A.10). We find

$$e_{(1,1,1)} = \frac{1}{6}(\mathbb{I} - (12) - (13) - (23) + (123) + (132)).$$

The frame  $\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}$  has two standard tableaux. For the tableaux  $\begin{smallmatrix} \boxed{1} & \boxed{2} \\ \boxed{3} \end{smallmatrix}$ ,  $\begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix}$ ,  $\begin{smallmatrix} \boxed{2} & \boxed{1} \\ \boxed{3} \end{smallmatrix}$ ,  $\begin{smallmatrix} \boxed{2} & \boxed{3} \\ \boxed{1} \end{smallmatrix}$ ,  $\begin{smallmatrix} \boxed{3} & \boxed{1} \\ \boxed{2} \end{smallmatrix}$ ,  $\begin{smallmatrix} \boxed{3} & \boxed{2} \\ \boxed{1} \end{smallmatrix}$  we find:

$$\begin{aligned} e_{(2,1)} &= \frac{2^2}{(3!)^2} \{ (\mathbb{I} + (12))(\mathbb{I} - (13)) + (\mathbb{I} + (13))(\mathbb{I} - (12)) + (\mathbb{I} + (12))(\mathbb{I} - (23)) + \\ &\quad + (\mathbb{I} + (23))(\mathbb{I} - (12)) + (\mathbb{I} + (13))(\mathbb{I} - (23)) + (\mathbb{I} + (23))(\mathbb{I} - (13)) \} \\ &= \frac{1}{3} \{ 2\mathbb{I} - (123) - (132) \}. \end{aligned}$$

Up to order  $p = 4$  the central idempotents are given by the sum of minimal projectors of the standard tableaux – they are orthogonal.

The characters in the irreducible  $(m)$  representation can be computed through

$$\chi_{(m)}(s) = \frac{f_{(m)}}{p!} \sum_{T|(m)(T)=(m)} \sum_{\substack{p \in \mathcal{R}(T) \\ q \in \mathcal{C}(T) \\ pq=s}} \text{sgn}(q).$$

Many other useful formulas can be derived from the Frobenius character formula. Interchanging rows and columns in a frame  $(m)$  leads us to the *dual frame*  $\widetilde{(m)}$ . For the characters one finds:  $\chi_{\widetilde{(m)}}(s) = \text{sgn}(s)\chi_{(m)}(s)$ . There is a nice formula for the characters of the transpositions in [FH91]:

Define the rank  $r$  of a frame to be the length of the diagonal. Let  $a_i$  and  $b_i$  be number of boxes below and to the right of the  $i$ 'th box, reading from lower right to upper left. Call  $\begin{pmatrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_r \end{pmatrix}$  the characteristics of  $(m)$ , e.g.

X				
	X			
		X		

$$r = 3, \text{ characteristics} = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 5 \end{pmatrix}.$$

Then

$$\chi_{(m)}(\tau) = \frac{f_{(m)}}{p(p+1)} \sum_{i=1}^r (b_i(b_i+1) - a_i(a_i+1)).$$

## APPENDIX B

### Notations and Abbreviations

$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	Minkowski metric
$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	Spinor metric
$\mathbb{N}, \mathbb{R}, \mathbb{C}$	Numbers: positive integer, real, complex
$\mathbb{M}$ Minkowski space	
$\mathcal{D}, \mathcal{S}, \mathcal{C}^\infty$	Test function spaces: compact support, rapid decrease, infinitely differentiable
$\mathfrak{A}, \mathfrak{B}, \mathfrak{G}, \mathfrak{G}_g$	Algebras: free quantized field, Boas algebra of symbols, sub algebras of generators and basic generators.
$\mathcal{H}, \mathcal{D}, \mathcal{F}$	Hilbert spaces: Hilbert space, dense domain of definition, Fock space
$\mathcal{L}, \mathcal{K}$	Lagrangian
$\widehat{f}(p) = \int dx f(x) e^{ipx}$	Fourier transformation in $\mathbb{R}^n$
$f(x) = \frac{1}{(2\pi)^n} \int dx \widehat{f}(x) e^{-ipx}$	
$\Delta, \Delta^F$	Commutator function, Feynman propagator
$D, D^F$	Massless commutator function, massless Feynman propagator
$\phi^{\text{class}}$	Classical field $\in \mathcal{D}(\mathbb{M})$
$\varphi_j$	Symbol $\in \mathfrak{G} \subset \mathfrak{B}$
$T(N)(x_N)$	Time ordered product
$R(M; N)(y_M; x_N)$	Retarded product
$N = \{W_1, \dots, W_n\}$	Set of symbols $W_i \in \mathfrak{B}$
$x_N = (x_1, \dots, x_n)$	$n$ -fold coordinate vector $\in \mathbb{M}^n$
$W_{g\mathcal{L}}$	Interacting field with local interaction $g\mathcal{L}, g \in \mathcal{D}(\mathbb{M}), \mathcal{L} \in \mathfrak{B}$ , in case of more couplings $g\mathcal{L} = \sum_i g_i \mathcal{L}_i$ .



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